Menger’s Theorem for directed graphs

Given \( x, y \in V(D) \), a set \( S \subseteq V(D) \setminus \{x, y\} \) is an \( x, y \)-separator (or an \( x, y \)-cut) if \( D - S \) has no \( x, y \)-path. Define

\[
\kappa_D(x, y) := \min\{|S| : S \text{ is an } x, y \text{-cut,}\} \quad \text{and} \quad \lambda_D(x, y) := \max\{|P| : P \text{ is a set of p.i.d. } x, y \text{-paths}\}
\]

**Directed-Local-Vertex-Menger Theorem** Let \( x, y \in V(D) \), such that \( xy \notin E(D) \). Then

\[
\kappa_D(x, y) = \lambda_D(x, y).
\]

**Proof. (Aharoni)** Let \( A = N^+(x) \) and \( B = N^-(y) \).

\[
D' := D - \{x, y\} - \{za : a \in A, z \in V(D)\} - \{bz : b \in B, z \in V(D)\}
\]

\( D' \): family of all \( A, B \)-paths in \( D' \).
**GOAL:** Find a family $\mathcal{P} \subseteq \mathcal{D}$ of pairwise disjoint $A, B$-paths and a subset $S \subseteq V(D')$ such that $|S \cap V(P)| \geq 1$ for every $P \in \mathcal{D}$ and $|S \cap V(P)| = 1$ for every $P \in \mathcal{P}$.

Proving the GOAL is indeed enough. (Think it over)

**Proof of GOAL.** Define an auxiliary bipartite graph $H$.

$V(H) := \{v^-, v^+: v \in V(D')\}$

$E(H) := \{u^+v^- : uv \in E(D')\} \cup \{v^-v^+: v \in V(D') \setminus A \setminus B\}$

By König’s Theorem there is a matching $M$ and a vertex-cover $C$ in $H$, such that $|e \cap C| = 1$ for every $e \in M$.

$\mathcal{P} := \{x_1 \cdots x_k \in \mathcal{D} : x_i^+x_{i+1}^- \in M \text{ for } 1 \leq i < k\}.$

$S := \{v \in V(D') : v^+, v^- \in C \text{ or } v^+ \in A^+ \cap C \text{ or } v^- \in B^- \cap C\}$.
• Any two paths $P_1, P_2 \in \mathcal{P}$ are disjoint.

$$V(P_1) \cap V(P_2) \neq \emptyset$$ implies there is $f_1 \in E(P_1)$, $f_2 \in E(P_2)$ such that $f_1 \neq f_2$ and $f_1 \cap f_2 \neq \emptyset$. $P_1, P_2 \in \mathcal{P}$ implies that for any $f_i \in E(P_i)$ either $f_1 = f_2$ or $f_1 \cap f_2 = \emptyset$.

• Any $A, B$-path $x_0x_1x_2 \cdots x_k$ contains a vertex from $S$.

Let $i$ be the largest index such that $x_i^- \in C$. (There is such, unless $x_0^+ \in C$ and $i < k$ unless $x_k^- \in C$)

Then $x_i^+ \in C$ since $x_i^+ x_{i+1}^-$ must be covered.

• No $A, B$-path $u_0u_1u_2 \cdots u_k = P \in \mathcal{P}$ contains more than one vertices from $S$.

Suppose $P$ does contain more. Let $u_i$ and $u_j \in S \cap V(P)$ such that $u_k \notin S$ for $i < k < j$. Then $u_i^+, u_j^- \in C$ by definition of $S$. Let $k$ be the largest index, $i < k < j$, such that $u_k^+ \in C$. Then $u_k^- \in C$ to cover the edge $u_k^- u_{k+1}^+$. Hence edge $u_k^+ u_{k+1}^- \in M$ is covered twice by $C$, a contradiction.
Corollaries

**Corollary** (Directed-Global-Vertex-Menger Theorem)
A digraph $D$ is strongly $k$-connected iff for any two vertices $x, y \in V(D)$ there exist $k$ p.i.d. $x, y$-paths.

*Proof: Lemma.* For every $e \in E(D)$, $\kappa_D(G-e) \geq \kappa_D(G)-1$.

The proof of the very first, the original Menger Theorem (the Undirected-Local-Vertex version) is

**HOMEWORK !!!**

Derive implication DLVM $\Rightarrow$ ULVM
Directed Edge-Menger

Given $x, y \in V(D)$, a set $F \subseteq E(D)$ is an $x, y$-disconnecting set if $D - F$ has no $x, y$-path. Define

$$\kappa'_D(x, y) := \min\{|F| : F \text{ is an } x, y\text{-disconnecting set,}\}$$

$$\lambda'_D(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* x, y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

Directed-Local-Edge-Menger Theorem For all $x, y \in V(D)$,

$$\kappa'_D(x, y) = \lambda'_D(x, y).$$

Proof. Create directed line graph and apply DLVM.

Corollary (Directed-Global-Edge-Menger Theorem) Directed multigraph $D$ is strongly $k$-edge-connected iff there is a set of $k$ p.e.d. $x, y$-paths for any two vertices $x$ and $y$. 

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