THE LIST COLORING CONJECTURE

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Introduction - The LCC
Kernels and choosability
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Introduction and the list coloring conjecture
vertex coloring

- **k-coloring** of a graph $G$: labelling $f: V(G) \rightarrow S$ with $|S|=k$. The labels are called **colors**
- A k-coloring is called **proper** if adjacent vertices have different colors
- $G$ is **k-colorable** if it has a proper k-coloring
- Chromatic number:
  $$\chi(G) := \min\{k | G \text{ is } k\text{-colorable}\}$$
example

3-coloring

Proper 3-coloring

$\chi(C_5) = 3$
edge coloring

- k-edge-coloring of a graph $G$: labelling $f: E(G) \rightarrow S$ with $|S| = k$.
- A k-edge-coloring is called proper if incident edges have different colors.
- $G$ is k-edge-colorable if it has a proper k-edge-coloring.
- Chromatic index: $\chi'(G) := \min\{k | G \text{ is } k\text{-edge-colorable}\}$.
example

3-edge-coloring  Proper 3-edge-coloring

$\chi'(C_5) = 3$
The line graph $L(G)$ of a graph $G$ is the graph with vertex set $V(L(G))=E(G)$ and edge set $E(L(G))=\{\{e,f\} \mid e \text{ incident with } f\}$.
example
example
List coloring is a more general concept of coloring a graph. A graph $G$ is $n$-choosable if, given any set of $n$ colors for each vertex, we can find a proper coloring.
C_5 is not 2-choosable
Choice number:

\[ \text{ch}(G) := \min \{ n \mid G \text{ is } n\text{-choosable} \} \]

The list chromatic index of \( H \) is the choice number of \( \text{L}(H) \)

Clearly: \( \text{ch}(G) \geq \chi(G) \)
$\text{ch}(K_{3,3}) > \chi(K_{3,3})$
list coloring conjecture

\[
\text{ch}(G) = \chi(G) \text{ whenever } G \text{ is a line graph}
\]
Kernels and choosability
Consider a digraph $D=(V,E)$

Notation: $u \rightarrow v$ means that $(u,v) \in E$:

The outdegree of $v$ is $d^+(v) = |\{ u \mid v \rightarrow u\}|$

The closed neighborhood of $v$ is $N[v] = \{ u \mid v \rightarrow u \text{ or } v=u \}$

The underlying graph of $D$ is $G=(V,E')$ with $E' = \{ \{ u,v \} \mid u \rightarrow v \text{ or } v \rightarrow u \}$
A kernel of $D$ is an independent set $K \subseteq V$ s.t. \[ \forall v \in V \setminus K \text{ there exists an } u \in K \text{ with } v \rightarrow u \]

A kernel of $S \subseteq V$ is a kernel of the subdigraph induced by $S$.
example
(f:g)-choosable

Consider two functions \( f, g : V \to \mathbb{N} \)

\( G \) is (f:g)-choosable if, given any sets \( A_v \) of colors with \( f(v) = |A_v| \), we can choose subsets \( B_v \subseteq A_v \) with \( g(v) = |B_v| \) such that \( B_u \cap B_v = \emptyset \) whenever \( \{u, v\} \in E \)

Example: \( G \) is \( n \)-choosable if we take \( f(v) = n \) and \( g(v) = 1 \)
lemma 1

Bondy, Boppana and Siegel

Let $D$ be a digraph in which
i. every induced subdigraph has a kernel
ii. $f, g: V(D) \rightarrow \mathbb{N}$ are so, that $f(v) \geq \sum_{u \in N[v]} g(u)$
whenver $g(v) > 0$

Then $D$ is $(f:g)$-choosable
Proof:

- Induction on $\sum_{v \in V} g(v)$
- Given: sets $A_v$ with $|A_v| = f(v)$
- Goal: find $B_v$ with $|B_v| = g(v)$ and $B_u \cap B_v = \emptyset$ whenever $u, v$ adjacent
- Define $W := \{v \in V | g(v) > 0\}$
- Choose color $c \in \bigcup_{v \in W} A_v$
- Define $S := \{v \in V | c \in A_v\}$
- Let $K$ be a kernel of $S$
• Define functions $f', g': V(D) \rightarrow \mathbb{N}$

\[
g'(v) = \begin{cases} 
  g(v) - 1 & (v \in K) \\
  g(v) & (v \not\in K)
\end{cases}
\]

\[
f'(v) = |A_v \setminus \{c\}| = \begin{cases} 
  f(v) - 1 & (c \in A_v) \\
  f(v) & (c \not\in A_v)
\end{cases}
\]

• i) holds, check ii)

• We have that $\sum_{u \in N[v]} g'(u) < \sum_{u \in N[v]} g(u)$

\[
f'(v) \geq \sum_{u \in N[v]} g'(u)
\]
• Induction hypothesis \( \Rightarrow G \text{ is } (f':g')\)-choosable

• This means: \( \exists B_v' \subseteq A_v \{c\} \text{ with } |B_v'| = g'(v), \text{ s.t. } B_v' \cap B_v' = \emptyset \text{ if } u,v \text{ adjacent} \)

• Define \( B_v := \begin{cases} B_v ' \cup \{c\} & (v \in K) \\ B_v ' & (v \notin K) \end{cases} \)

• It holds: \( |B_v| = g(v) \) and \( B_u \cap B_v = \emptyset \text{ if } u,v \text{ adjacent} \)

• \( \Rightarrow G \text{ is } (f:g)\)-choosable
G is called \((m:n)\)-choosable if it is \((f:g)\)-choosable for the constant functions \(f(v)=m\) and \(g(v)=n\)
corollary 1

Let D be a digraph in which
i. the maximum outdegree is n-1
ii. every induced subdigraph has a kernel
Then D is (kn:k)-choosable for every k.
In particular D is n-choosable.
Proof of the bipartite LCC
An orientation of a graph \( G \) is any digraph having \( G \) as underlying graph.

Let \( K \) be a set of vertices in a digraph. Then \( K \) absorbs a vertex \( v \) if \( \text{N}[v] \cap K \neq \emptyset \).

\( K \) absorbs a set \( S \) if \( K \) absorbs every vertex of \( S \).

Example: a kernel of \( S \) is an independent subset of \( S \) that absorbs \( S \).
Let $H$ be a bipartite multigraph, and $G := L(H)$. Suppose that $G$ is $n$-colorable. Then $G$ is $(kn:k)$-choosable for every $k$. In particular $G$ is $n$-choosable.
Proof:

- Let $V := V(G) = E(H)$
- Define $A_x := \{v \in V | \ v \text{ is incident with } x, \ x \in V(H)\}$. $A_x$ is called row if $x \in X$, column if $x \in Y$
- If $v$ is a vertex of $G$, then $R(v)$ is the row and $C(v)$ the column containing $v$
- Take a proper coloring $f: V \rightarrow \{1, \ldots, n\}$ of $V$
- Define an orientation $D$ of $G$ in which $u \rightarrow v$ if $R(u)=R(v)$ and $f(u)>f(v)$ or $C(u)=C(v)$ and $f(u)<f(v)$
• Check i) and ii) of Corollary 1

• i): ✓ (since f is one-to-one on N[v])

• ii): induction on |S|

• Given S ⊆ V, show the existence of a kernel in S

• Define

\[ T := \{ v \in S \mid f(v) < f(u) \text{ whenever } v \neq u \in R(v) \cap S \} \]

• T absorbs S

• If T is independent, then it’s a kernel ✓
• Assume $T$ is not independent, then it has two elements in the same column, say $v_1, v_2 \in T$ with $C(v_1) = C(v_2) =: C$, $f(v_1) < f(v_2)$

• Choose $v_0 \in C \cap S$ s.t. $f(v_0) < f(u)$ whenever $v_0 \neq u \in C \cap S$

• Induction hypothesis $\Rightarrow S \setminus \{v_0\}$ has a kernel $K$, and $K$ absorbs $v_2$, this means $N[v_2] \cap K \neq \emptyset$

• $N[v_0] \cap K \supseteq N[v_2] \cap K \Rightarrow K$ absorbs $v_0$

$\Rightarrow K$ is a kernel of $S$
• Corollary 1 ⇒ $D$ is $(kn:k)$-choosable $\forall k$
• $\Rightarrow G$ is $(kn:k)$-choosable for every $k$