General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:
  
  **Group A/B:** Wed 13–15 CAB G 56
  **Group C:** Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer”, then a formal proof is always required.

The following exercises will be discussed in the exercise class on November 15, 2016. Please hand in your solutions not later than November 14.

**Exercise 1: Finding a Separating Line**

Let \( R, B \subseteq \mathbb{R}^2 \) be given finite sets (“red and blue points”). A \((\text{strictly})\) \textit{separating line} is a line \( \ell \) with the property that all red points lie strictly on one side of \( \ell \), and all blue points strictly on the other side.

Formulate a linear program such that, given an optimal solution of your LP, you can decide if a separating line exists and, if so, compute one.

**Exercise 2: Fitting a Ball into a Convex Polytope**

Let \( H_1, \ldots, H_m \) be halfspaces in \( \mathbb{R}^n \) given by \( H_i = \{x: a_i^T x \leq b_i\} \) where \( a_i \in \mathbb{R}^n \) and \( b_i \in \mathbb{R} \). We want to find the largest \( n \)-dimensional ball that is completely contained in the intersection \( \bigcap_{i=1}^m H_i \), which is assumed to be non-empty.

Formulate a linear program with variables \( c \in \mathbb{R}^n \) and \( r \in \mathbb{R} \) whose optimal solution is the center point \( c^* \) and radius \( r^* \) of this largest ball.
Exercise 3: Linear Programs in Equational Form

Show that every linear program can also be converted into the following *equational form*:

\[
\text{maximize } c^T x \text{ subject to } Ax = b, \quad x \geq 0.
\]

What is the maximum increase in the number of variables and in the number of constraints in such a transformation?

Exercise 4: Maximum Number of Vertices of 3-dimensional Convex Polytopes

Let \( P \) be a convex polytope which is defined as the intersection of \( n \) given closed half-spaces in \( \mathbb{R}^3 \). Show that the number of vertices of \( P \) is at most \( 2n - 4 \).

**Hint:** Use Euler’s formula for plane graphs, \( v - e + f = 2 \).

Exercise 5: Certificates for Infeasibility of Systems of Linear Equations

(Preparation for chapter 6.5.)

Prove that a system \( Ax = b \) of linear equations is unsolvable if and only if there is \( y \) with \( A^Ty = 0 \) and \( b^Ty = 1 \).