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Algorithms, Probability, and Computing **Solutions KW38** **HS17**

Solution 1: Dependency

(a) Since X_1 and X_2 are independent, we have

$$\begin{aligned}\Pr[X_1 + X_2 = 8] &= \sum_{x_1=1}^{6} \Pr[X_1 = x_1 \wedge X_2 = 8 - x_1] \\ &= \sum_{x_1=1}^{6} \Pr[X_1 = x_1] \cdot \Pr[X_2 = 8 - x_1] \\ &= 5 \cdot \frac{1}{36},\end{aligned}$$

where we have used that $\Pr[X_2 = 7] = 0$, while all other probabilities are $1/6$.

(b) By the definition of conditional probability, we have

$$\Pr[X_1 + X_2 \geq 6 \mid X_1 \leq 2] = \frac{\Pr[X_1 + X_2 \geq 6 \wedge X_1 \leq 2]}{\Pr[X_1 \leq 2]} =$$

$$= \frac{\Pr[X_1 = 1] \cdot \sum_{i=5}^{6} \Pr[X_2 = i] + \Pr[X_1 = 2] \cdot \sum_{i=4}^{6} \Pr[X_2 = i]}{\Pr[X_1 = 1] + \Pr[X_1 = 2]} = \frac{\frac{1}{6} \cdot \frac{2}{6} + \frac{1}{6}\frac{3}{6}}{\frac{1}{6} + \frac{1}{6}} = \frac{5}{12}.$$

(c) In each of the following cases, we calculate (by manually counting how many of the 36 possible outcomes are contained in the events) the values of $p_1 = \Pr[\mathcal{E}_1]$, $p_2 = \Pr[\mathcal{E}_2]$, and $p_{12} = \Pr[\mathcal{E}_1 \cap \mathcal{E}_2]$, respectively. The events are then independent iff $p_1 p_2 = p_{12}$.

(i)
- $p_1 = 1/2$.
- $p_2 = 1/2$.
- $p_{12} = 1/4$.
- independent

(ii)
- $p_1 = 1/2$.
- $p_2 = 5/12$.
- $p_{12} = 3/12$.
- dependent

(iii)
- $p_1 = 1/6$.
- $p_2 = 1/6$.
- $p_{12} = 1/18$.
- dependent

(iv)
- $p_1 = 7/12$.
- $p_2 = 1/12$.
- $p_{12} = 1/18$.
- dependent

Solution 2: Geometric Distributions

(a) Let X be the random variable that describes the number of runs until we encounter the first success. For example, abbreviating 'failure' by F and 'success' by S, if we encounter the sequence FFS then X would assume the value 3. The distribution of X is given by

$$\Pr[X = k] = (1 - p)^{k-1} p \quad (k = 1, 2, \dots).$$

As we remember from the course *Probability and statistics*,

$$\mathbf{E}[X] = \frac{1}{p}.$$

If we don't remember then we can also compute it like this:

$$\begin{aligned} \mathbf{E}[X] &= \sum_{k=1}^{\infty} \Pr[X \geq k] \\ &= \sum_{k=1}^{\infty} (1-p)^{k-1} \\ &= \frac{1}{1-(1-p)} \quad \text{(geometric series!)} \\ &= \frac{1}{p}. \end{aligned}$$

(b) We sum over all even values for X and obtain (using the standard formula for geometric series)

$$\Pr[X \text{ even}] = \sum_{j=1}^{\infty} \Pr[X = 2j] = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{2j} = \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{1}{4}\right)^{j} = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}.$$

(c) According to the same calculation,

$$\Pr[X \text{ even}] = \sum_{j=1}^{\infty} \Pr[X = 2j] = \sum_{j=1}^{\infty} (1-p)^{2j-1} p$$
$$= p(1-p) \sum_{j=0}^{\infty} \left((1-p)^2\right)^j = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

Solution 3: Expected running time

(a) Applying the definition of expected value,

$$\mathbf{E}[X] = \sum_{x=1}^{3} \Pr[X = x] \cdot x = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 = \frac{7}{4}.$$

Likewise,

$$\mathbf{E}\left[X^2\right] = \sum_{x=1}^{3} \Pr[X = x] \cdot x^2 = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 9 = \frac{15}{4}.$$

Thus we are reminded that the numbers $\mathbf{E}\left[X^2\right]$ and $\mathbf{E}[X]^2$ are, in general, not equal. Indeed our example has $\mathbf{E}\left[X^2\right] = \frac{15}{4}$, but $\mathbf{E}[X]^2 = \frac{49}{16}$.

(b) The random variables X_1, X_2 are independent and have the same distribution as X. (Note: This is only true because of the specific way the question is phrased. In general we have to be careful whether our random variables are really independent.)

For (i), by applying linearity of expectation, we get

$$\mathbf{E}[X_1 + X_2] = \mathbf{E}[X_1] + \mathbf{E}[X_2] = 2\,\mathbf{E}[X] = \frac{7}{2}.$$

Note that this would hold true even if X_1 and X_2 were dependent. For (ii), on the other hand, we use that X_1,X_2 are independent, and then

$$\mathbf{E}[X_1 \cdot X_2] = \mathbf{E}[X_1] \cdot \mathbf{E}[X_2] = \mathbf{E}[X]^2 = \frac{49}{16}.$$

For (iii), to get a sum of $X_1 + X_2 \leq 4$, there are only the following possibilities:

- $X_1 = 1$,
- $X_1 = 2$ and $X_2 \in \{1, 2\}$, or
- $X_1 = 3$ and $X_2 = 1$.

Since these three events are disjoint, we find

$$\Pr[X_1 + X_2 \leq 4] = \Pr[X_1 = 1] + \Pr[X_1 = 2 \text{ and } X_2 \in \{1,2\}] + \Pr[X_1 = 3 \text{ and } X_2 = 1].$$

Since X_1 and X_2 are independent, we obtain

$$\begin{aligned}\Pr[X_1 + X_2 \leq 4] &= \Pr[X_1 = 1] + \Pr[X_1 = 2] \cdot \Pr[X_2 \in \{1,2\}] + \Pr[X_1 = 3] \cdot \Pr[X_2 = 1] \\ &= \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{13}{16}.\end{aligned}$$

(c) It might seem as if the running time is described by the random variable $X \cdot N$, which would lead to the result $\mathbf{E}[X] \cdot \mathbf{E}[N]$ as long as X and N are independent.

However, it is important to note that $X \cdot N$ does *not* describe our situation. It would only be correct to use $X \cdot N$ if, for some reason, every subroutine call had the exact same running time. (Why?)

The correct way to express the overall running time is to use a sequence of random variables $X_1, \ldots, X_N$, where X_i describes the running time of the ith subroutine call. In order to be able to compute with the strange formula $X_1 + \cdots + X_N$, we actually use an infinite sequence of variables $X_1, X_2, \ldots$, where the variable X_i is defined to assume the value 0 whenever $i > N$.

We then have, for all $i \geq 1$:

$$\begin{aligned}\mathbf{E}[X_i \,|\, i \leq N] &= \mathbf{E}[X], \\ \mathbf{E}[X_i \,|\, i > N] &= 0.\end{aligned}$$

Now we can calculate

$$\begin{aligned}\mathbf{E}[X_1 + \ldots X_N] &= \mathbf{E}\left[\sum_{i=1}^{\infty} X_i\right] \\ &= \sum_{i=1}^{\infty} \mathbf{E}[X_i] \quad \text{(by montone convergence)} \\ &= \sum_{i=1}^{\infty} \Big(\underbrace{\mathbf{E}[X_i \mid i \leq N]}_{\mathbf{E}[X]} \cdot \Pr[i \leq N] + \underbrace{\mathbf{E}[X_i \mid i > N]}_{0} \cdot \Pr[i > N] \Big) \\ &= \mathbf{E}[X] \cdot \sum_{i=1}^{\infty} \Pr[i \leq N] \\ &= \mathbf{E}[X] \cdot \mathbf{E}[N].\end{aligned}$$

Solution 4: Random Walks

(a) For any $v \in \{A, B, C, D, E\}$, let us write e_v to denote the expected number of days needed to reach vertex A given that the worm starts from vertex v. The value we are looking for in this task is e_C.

When starting from vertex C, the worm has a probability of 1/2 to go to B in the first step, and a probability of 1/2 to go to D. If it reaches B, it needs another e_B number of days on average to reach A. If it reaches D, it needs e_D expected number of days. Therefore

$$e_C = \frac{1}{2}e_B + \frac{1}{2}e_D + 1.$$

We can write analogous relations for the other quantities:

$$\begin{aligned} e_A &= 0, \\ e_B &= \frac{1}{2}e_A + \frac{1}{2}e_C + 1, \\ e_D &= \frac{1}{2}e_C + \frac{1}{2}e_E + 1, \\ e_E &= \frac{1}{2}e_D + \frac{1}{2}e_A + 1. \end{aligned}$$

This way, we have a linear system of five equations and five unknowns that we can solve. The result is that $e_C = 6$.

(b) According to Markov's inequality, we have

$$\Pr[T \geq 100] = \Pr\left[T \geq \frac{100}{6}e_c\right] \leq \frac{6}{100}.$$

The probability for the worm to take at least 100 days until dinner is at most 6%.

Solution 5: Independence of Three Events

Recall that the events A, B, C are called *pairwise independent* if they satisfy

$$\begin{aligned} \Pr[A \cap B] &= \Pr[A] \cdot \Pr[B], \\ \Pr[A \cap C] &= \Pr[A] \cdot \Pr[C], \\ \Pr[B \cap C] &= \Pr[B] \cdot \Pr[C]; \end{aligned}$$

and they are called *mutually independent* if **in addition** they satisfy

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C].$$

A typical example would be tossing two fair coins. Let A be the event that the first coin lands head. Let B be the event that the second coin lands head. And let C be the event that the two coin tosses land the same. Each event has a probability of 1/2. By calculating

$$\Pr[A \cap B] = \frac{1}{4},$$
$$\Pr[A \cap C] = \frac{1}{4},$$
$$\Pr[B \cap C] = \frac{1}{4},$$

we see that A, B, C are pairwise independent. But $\Pr[A \cap B \cap C] = 1/4$, not $1/8$ as we would expect for jointly independent events.

Solution 6: Conditional Probability

(a) Intuition: the event that the egg is spoiled is completely independent of the event that the milk is spoiled. Therefore the probability that the egg is spoiled is not influenced by the information that the milk is spoiled.

Formally: Let E be the event that the egg is spoiled and M the event that the milk is spoiled. We are interested in the probability $\Pr[E|M]$. We have

$$\Pr[E|M] = \frac{\Pr[E \cap M]}{\Pr[M]} = \frac{1/4}{1/2} = \frac{1}{2}.$$

(b) Intuition: having exactly one boy and exactly one girl is more likely (1/2) than having two boys (1/4). Thus if we know that there is at least one boy, it is more likely for the other child to be a girl than that both are boys. Note the important difference to the situation in (a). There, the information we were conditioning on concerned exactly one of the two experiments ("the milk").

In this case, the information concerns both experiments jointly ("one of the two is"). If the information given were that the older child is a boy, then the probability to get another boy would not be influenced by it (given our simplifying assumption on independence).

Formally: Let B be the event that at least one child is a boy and C the event that both children are boys. We are interested in $\Pr[C|B]$.

$$\Pr[C|B] = \frac{\Pr[C \cap B]}{\Pr[B]} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Solution 7: Paradoxes

(a) We model steps 1 and 2 (but not step 3) of the described game show as a probability space $\Omega = \{CG_1, CG_2, G_1G_2, G_2G_1\}$. The meaning of the four elementary events is as follows.

CG_1 — You point at the car, and the show master reveals goat number 1.

CG_2 — You point at the car, and the show master reveals goat number 2.

G_1G_2 — You point at goat number 1, and the show master reveals goat number 2.

G_2G_1 — You point at goat number 2, and the show master reveals goat number 1.

In step 1 of the game show, you choose either (i) the car, or you choose (ii) one of the two goats with probabilty $1/3$ each. In step 2, the show master reveals (i) one of the two goats with probabilty $1/2$ each, or he reveals (ii) the unique remaining goat with probability 1. Hence, it is not hard to see that the four elementary events must have the following probabilities.

$$\Pr[CG_1] = \Pr[CG_2] = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \qquad \Pr[G_1G_2] = \Pr[G_2G_1] = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

We define two additional events. Let S be the event that switching in step 3 would turn out to be beneficial to you. More formally, $S = \{G_1G_2, G_2G_1\}$ is the event that in step 1 you point at one of the two goats (or, more precisely, at a door behind which there is a goat). Furthermore[1], let R be the event that the show master reveals a goat in step 2. By inspecting the list of elementary events, or simply by rereading the rules, we see that trivially $R = \Omega$. In other words, the show master *always* reveals a goat in step 2 no matter what happens in step 1.

All that is left to do for a complete solution is to ask the right question. Given that the show master has revealed a goat in step 2, how likely is it that you would win the car if you were to switch your chosen door in step 3?

$$\Pr[S \mid R] = \frac{\Pr[S \cap R]}{\Pr[R]} = \frac{\Pr[S]}{\Pr[R]} = \frac{2/3}{1} = \frac{2}{3}$$

In the concrete situation described on the exercise sheet, it is therefore advisable to switch from door 1 to door 3 since that gives a winning chance of $2/3$.

As far as intuition is concerned, observe that conditioning on R does not change the probability of any event that is expressible as a subset of Ω because R is a certain event[2] in that space. In particular, the probability that the car is behind the door chosen in step 1 remains $1/3$ even after the show master reveals one of the goats, which in turn means that the car is behind the third door with probability $2/3$. In other words, when the show master reveals a goat in step 2 there is absolutely no gain of information with respect to our model Ω.

(b) Assume that goat number 1 is black and that goat number 2 is white. Compared with the previous task, the probabilities of the four elementary events have changed as follows.

$$\Pr[CG_1] = \frac{p}{3} \qquad \Pr[CG_2] = \frac{1-p}{3} \qquad \Pr[G_1G_2] = \Pr[G_2G_1] = \frac{1}{3}$$

[1]Defining this second event R is not really necessary for a complete solution of task (a). However, it is key in understanding the difference between tasks (a) and (c).

[2]dt. Sicheres Ereignis

Let us further define the events $R_1 = \{CG_1, G_2G_1\}$ and $R_2 = \{CG_2, G_1G_2\}$ that the show master reveals the black goat or the white goat, respectively, in step 2. As follows, we can now calculate the corresponding winning chances for switching in step 3 for each revealed color goat individually.

$$\Pr[S \mid R_1] = \frac{\Pr[S \cap R_1]}{\Pr[R_1]} = \frac{\Pr[G_2G_1]}{\Pr[CG_1] + \Pr[G_2G_1]} = \frac{1/3}{p/3 + 1/3} = \frac{1}{1+p}$$

$$\Pr[S \mid R_2] = \frac{\Pr[S \cap R_2]}{\Pr[R_2]} = \frac{\Pr[G_1G_2]}{\Pr[CG_2] + \Pr[G_1G_2]} = \frac{1/3}{(1-p)/3 + 1/3} = \frac{1}{2-p}$$

For any value of p, the above numbers are at least $1/2$, which means that switching in step 3 is always at least as good as sticking with the original choice. In fact, unless $p = 0$ or $p = 1$, switching is even the strictly better option.

If $p = 0$ then $\Pr[S \mid R_2] = 1/2$. That is, if the show master always reveals the white goat if he has the choice, and you actually see him revealing the white goat in step 2 of the game show, then switching and sticking with the original choice are both equally good. A similar thing can be said if $p = 1$ and you see the black goat revealed in step 2.

(c) Similar to what we did in task (a), we model the situation up to the point where Hermione chooses and drinks one of the cups as a probability space. Here, we have the following set $\Omega = \{GP_1, GP_2, P_1G, P_1P_2, P_2G, P_2P_1\}$ of elementary events.

GP_1 — Harry picks the good potion, Hermione drinks poison number 1.

GP_2 — Harry picks the good potion, Hermione drinks poison number 2.

P_1G — Harry picks poison number 1, Hermione drinks the good potion.

P_1P_2 — Harry picks poison number 1, Hermione drinks poison number 2.

P_2G — Harry picks poison number 2, Hermione drinks the good potion.

P_2P_1 — Harry picks poison number 2, Hermione drinks poison number 1.

Given that all choices are made uniformly at random, we arrive at the following probabilities.

$$\Pr[GP_1] = \Pr[GP_2] = \Pr[P_1G] = \Pr[P_1P_2] = \Pr[P_2G] = \Pr[P_2P_1] = \frac{1}{6}$$

Let now $S = \{P_1P_2, P_2P_1\}$ be the event that switching his choice is beneficial to Harry after Hermione has drunk her cup. Furthermore, let $R = \{GP_1, GP_2, P_1P_2, P_2P_1\}$ be the event that Hermione reveals one of the 2 poisoned cups by dying after drinking from such a cup. It is crucial to note that here, in contrast to task (a), we do not have $R = \Omega$. In other words, Hermione does *not always* reveal a poisoned cup.

We are left again with asking the right question. Given that Hermione has died after drinking from her cup, how likely is it that Harry would drink the good potion if he were to switch his choice to the unclaimed third cup?

$$\Pr[S \mid R] = \frac{\Pr[S \cap R]}{\Pr[R]} = \frac{\Pr[S]}{\Pr[R]} = \frac{2/6}{4/6} = \frac{1}{2}$$

The surprising answer here is therefore that it does not matter if Harry sticks with his first choice or not. After Hermione has died from drinking one of the cups, the two remaining cups hold the good potion with probability $1/2$ each.

Observe that here we have $R \neq \Omega$, which means that conditioning on R has the potential of changing the probability of any event that is expressible as a subset of Ω. In particular, the probability of Harry having picked the good potion in the beginning increases from $1/3$ to $1/2$. Intuitively, when Hermione dies we get the information that she was unlucky and did not manage to pick the good potion, which makes it more likely that Harry is holding it already in his hand.