

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

Group A: Wed 13–15 CAB G 56

Group B: Wed 13–15 CHN D 44

Group C: Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer”, then a formal proof is **always** required.
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The following exercises will be discussed in the exercise classes on September 26, 2018. Please hand in your solutions in the lecture, no later than September 25.

Exercise 1

Let $G = (V, E)$ be a connected graph with weights $w : E \rightarrow \mathbb{R}$ on the edges, and define the edge boundary of set S to be

$$\partial(S) := \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$$

Assume that for every non-empty vertex set $S \subset V$, the edge with the minimum weight in $\partial(S)$ is unique.

Prove that G has a unique MST. Conclude that if the weight function w is injective (i.e., no two edges have the same weight), G contains exactly one MST.

Exercise 2

You already know that for a connected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, the expected running time of *Randomized Minimum Spanning Tree Algorithm* (G) (see page 5 in the lecture notes) is equal to $\mathcal{O}(m)$.

(i) Prove that the worst-case running time of the algorithm is equal to $\mathcal{O}(\min\{n^2, m \log n\})$.

(ii) Prove that the running time of the algorithm is equal to $\mathcal{O}(m)$ with probability $1 - o(1)$.

Hint: You might apply the Chernoff bound (see the help sheet on the website of the course).

Exercise 3

For a graph $G = (V, E)$, a cut is the partition of the vertex set V into two disjoint sets V_1 and V_2 and the size of the cut is the number of edges between V_1 and V_2 .

(i) Assume that $|V|$ is even. We say a cut is balanced if $|V_1| = |V_2| = |V|/2$. Prove that there always exists a balanced cut of size at least $|E|/2$.

Hint: Since you want to prove the existence, you might apply randomness.

(ii) Define $d_S(v)$ to be the number of neighbors of vertex v in a set $S \subseteq V$, i.e. $d_S(v) := |\{u \in S : \{v, u\} \in E\}|$. Now, consider the following algorithm. Partition the vertex set V into two arbitrary sets V_1 and V_2 . As far as there is a vertex $v \in V_1$ (or $v \in V_2$) for which $d_{V_1}(v) > d_{V_2}(v)$ (resp. $d_{V_2}(v) > d_{V_1}(v)$) move v from V_1 to V_2 (resp. from V_2 to V_1). Prove that this algorithm terminates and generates a cut of size at least $|E|/2$.

Exercise 4

In a city there are n houses h_1, \dots, h_n , each of which is in need of a water supply. It costs c_i to build a well at house h_i , and it costs w_{ij} to build a pipe in between houses h_i and h_j . A house can receive water if either there is a well built there or there is some path of pipes to a house with a well. Give an algorithm to find the minimum cost to supply every house with water.

Hint: There is a short solution.