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Algorithms, Probability, and Computing **Solutions KW44** **HS18**

Solution 1

Let us first do the easy direction. Suppose there is $\mathbf{y}$ with $A^\mathsf{T}\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} = 1$. Furthermore, towards a contradiction, suppose there is an $\mathbf{x}$ with $A\mathbf{x} = \mathbf{b}$ (or, equivalently, $\mathbf{x}^\mathsf{T}A^\mathsf{T} = \mathbf{b}^\mathsf{T}$). We arrive at a contradiction (and hence conclude that $A\mathbf{x} = \mathbf{b}$ is unsolvable) by observing that

$$0 = \mathbf{x}^\mathsf{T}\mathbf{0} = \mathbf{x}^\mathsf{T}A^\mathsf{T}\mathbf{y} = \mathbf{b}^\mathsf{T}\mathbf{y} = 1.$$

For the other direction we recall some notation from linear algebra (we assume throughout that the matrix A has m rows and n columns). The *image* of A is the set $\mathrm{img}(A) := \{A\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^n\}$. The *left nullspace* (or *cokernel*) of A is the set $\ker(A^\mathsf{T}) := \{\mathbf{y} \in \mathbf{R}^m \mid A^\mathsf{T}\mathbf{y} = \mathbf{0}\}$. We also recall that these two sets are vector spaces and that they are orthogonal complements of each other. In particular, if $\mathbf{i}_1, \ldots, \mathbf{i}_r$ is an orthonormal basis of $\mathrm{img}(A)$ and $\mathbf{k}_1, \ldots, \mathbf{k}_s$ is an orthonormal basis of $\ker(A^\mathsf{T})$, then $\mathbf{i}_1, \ldots, \mathbf{i}_r, \mathbf{k}_1, \ldots, \mathbf{k}_s$ is an orthonormal basis of $\mathbf{R}^m$.

Now suppose that the system $A\mathbf{x} = \mathbf{b}$ is unsolvable. We show how to construct $\mathbf{y}$ with $A^\mathsf{T}\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} = 1$. First we write $\mathbf{b}$ as a linear combination $\mathbf{b} = \alpha_1\mathbf{i}_1 + \cdots + \alpha_r\mathbf{i}_r + \beta_1\mathbf{k}_1 + \cdots + \beta_s\mathbf{k}_s$. We observe that $s \geq 1$ and that for some index i we must have $\beta_i \neq 0$ (for otherwise $\mathbf{b} \in \mathrm{img}(A)$, which cannot be if $A\mathbf{x} = \mathbf{b}$ is unsolvable). W.l.o.g. we assume that $\beta_1 \neq 0$ and we define $\mathbf{y} := \frac{1}{\beta_1}\mathbf{k}_1$. We now see that $A^\mathsf{T}\mathbf{y} = \mathbf{0}$ because $\mathbf{y} \in \ker(A^\mathsf{T})$. Moreover,

$$\mathbf{b}^\mathsf{T}\mathbf{y} = \frac{\alpha_1}{\beta_1}\underbrace{\mathbf{i}_1^\mathsf{T}\mathbf{k}_1}_{=0} + \cdots + \frac{\alpha_r}{\beta_1}\underbrace{\mathbf{i}_r^\mathsf{T}\mathbf{k}_1}_{=0} + \frac{\beta_1}{\beta_1}\underbrace{\mathbf{k}_1^\mathsf{T}\mathbf{k}_1}_{=1} + \frac{\beta_2}{\beta_1}\underbrace{\mathbf{k}_2^\mathsf{T}\mathbf{k}_1}_{=0} + \cdots + \frac{\beta_s}{\beta_1}\underbrace{\mathbf{k}_s^\mathsf{T}\mathbf{k}_1}_{=0} = \frac{\beta_1}{\beta_1} = 1.$$

Solution 2

(a) Suppose there is $\mathbf{y}$ with $A^\mathsf{T}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$. Furthermore, towards a contradiction, suppose there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} = \mathbf{b}$ (or, equivalently, $\mathbf{x}^\mathsf{T}A^\mathsf{T} = \mathbf{b}^\mathsf{T}$). We arrive at a contradiction (and hence conclude that $A\mathbf{x} = \mathbf{b}$ has no non-negative solution) by observing that

$$0 \leq \mathbf{x}^\mathsf{T}A^\mathsf{T}\mathbf{y} = \mathbf{b}^\mathsf{T}\mathbf{y} < 0,$$

where the first inequality is justified because both $\mathbf{x}^\mathsf{T}$ and $A^\mathsf{T}\mathbf{y}$ are non-negative.

Now suppose instead that there is $\mathbf{y} \geq \mathbf{0}$ with $A^\mathsf{T}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$. Furthermore, towards a contradiction, suppose there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} \leq \mathbf{b}$ (or, equivalently,

$x^TA^T \le \mathbf{b}^T$). We arrive at a contradiction (and hence conclude that $Ax \le \mathbf{b}$ has no non-negative solution) by observing that

$$0 \le x^TA^T\mathbf{y} \le \mathbf{b}^T\mathbf{y} < 0,$$

where the second inequality is justified because of our assumption $Ax \le \mathbf{b}$ and because $\mathbf{y}$ is non-negative.

(b) We only prove the implication I $\Rightarrow$ II. The other implications can be proved in a very similar fashion.

We note that $Ax = \mathbf{b}$ has no non-negative solution x if and only if the system $\mathcal{A}x \le \mathcal{B}$ is unsolvable, where

$$\mathcal{A} := \begin{pmatrix} A \\ -A \\ -\mathbb{1} \end{pmatrix}, \qquad \mathcal{B} := \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{pmatrix},$$

and where $\mathbb{1}$ is the identity matrix of appropriate dimension. Indeed, the system $\mathcal{A}x \le \mathcal{B}$ just encodes the constraints $Ax \le \mathbf{b}$, $Ax \ge \mathbf{b}$ and $x \ge \mathbf{0}$. We now use Farkas lemma I and see that $\mathcal{A}x \le \mathcal{B}$ is unsolvable if and only if there is a vector $\mathcal{Y} \ge \mathbf{0}$ with $\mathcal{A}^T\mathcal{Y} = \mathbf{0}$ and $\mathcal{B}^T\mathcal{Y} < 0$. We write the vector $\mathcal{Y}$ as

$$\mathcal{Y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{pmatrix}$$

so that $\mathcal{A}^T\mathcal{Y} = A^T\mathbf{y}_1 - A^T\mathbf{y}_2 - \mathbf{y}_3$ and $\mathcal{B}^T\mathcal{Y} = \mathbf{b}^T\mathbf{y}_1 - \mathbf{b}^T\mathbf{y}_2$. Finally, we note that there exists $\mathcal{Y} \ge \mathbf{0}$ with $\mathcal{A}^T\mathcal{Y} = \mathbf{0}$ and $\mathcal{B}^T\mathcal{Y} < 0$ if and only if there is $\mathbf{y}$ with $A^T\mathbf{y} \ge 0$ and $\mathbf{b}^T\mathbf{y} < 0$, which concludes the proof. Indeed, for the "only if" we define $\mathbf{y} := \mathbf{y}_1 - \mathbf{y}_2$ and see that

$$A^T\mathbf{y} = A^T\mathbf{y}_1 - A^T\mathbf{y}_2 \ge A^T\mathbf{y}_1 - A^T\mathbf{y}_2 - \mathbf{y}_3 = \mathcal{A}^T\mathcal{Y} = \mathbf{0},$$

where the inequality is justified because $\mathcal{Y}$ (and thus also $\mathbf{y}_3$) is non-negative, and we also see that

$$\mathbf{b}^T\mathbf{y} = \mathbf{b}^T\mathbf{y}_1 - \mathbf{b}^T\mathbf{y}_2 = \mathcal{B}^T\mathcal{Y} < \mathbf{0}.$$

For the "if" we can always choose $\mathbf{y}_1$ and $\mathbf{y}_2$ in such a way that both are non-negative and such that $\mathbf{y}_1 - \mathbf{y}_2 = \mathbf{y}$. Since we know that $A^T\mathbf{y} = A^T\mathbf{y}_1 - A^T\mathbf{y}_2 \ge \mathbf{0}$ we can also choose a non-negative $\mathbf{y}_3$ with $A^T\mathbf{y} - A^T\mathbf{y}_2 - \mathbf{y}_3 = \mathbf{0}$.

Solution 3

(a) Every $x \in \{0,1\}^n$ is a basic feasible solution since n inequalities hold with equality and they are clearly linearly independent (the identity matrix is non-singular). These are all basic feasible solutions since in order to get n equalities we have to set for every $i \in [n]$ either $x_i = 0$ or $x_i = 1$. Let $I \subseteq [n]$ be the set of all indices $i \in [n]$ so that $c_i < 0$. Then clearly an optimum basic feasible solution $\tilde{x}$ is given by setting $\tilde{x}_i = 1$ if $i \in I$ and $\tilde{x}_i = 0$ otherwise.

(b) Any basic feasible solution has to satisfy the equality $\mathbf{1}^\mathsf{T}\mathbf{x} = 1$ and $n-1$ coordinates of $\mathbf{x}$ have to be zero. Therefore the remaining coordinate has to be 1. This means that the basic feasible solutions are $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, i.e., all standard unit vectors. Let $i = \arg\min_{j\in[n]} \mathbf{c}_j$. Clearly the objective function is minimized by $\mathbf{e}_i$.

(c) If the constraint $\mathbf{1}^\mathsf{T}\mathbf{x} \leq k$ does not hold with equality, then we are back to case (a) with the additional constraint that $\mathbf{x}$ should have at most $k-1$ coordinates set to 1, i.e., such basic feasible solutions form the set $\{\mathbf{x} \in \{0,1\}^n \mid \mathbf{1}^\mathsf{T}\mathbf{x} \leq k-1\}$. If $\mathbf{1}^\mathsf{T}\mathbf{x} = k$, then in any basic feasible solution $n-1$ additional inequalities have to hold as equalities. Therefore $n-1$ of the coordinates in $\mathbf{x}$ are either 0 or 1 and only one coordinate is allowed to be fractional. But since $\mathbf{1}^\mathsf{T}\mathbf{x} = k$ and since k is an integer, the last coordinate has to be also either 0 or 1. All in all we get that the set of basic feasible solutions is $\{\mathbf{x} \in \{0,1\}^n \mid \mathbf{1}^\mathsf{T}\mathbf{x} \leq k\}$.

Finding the basic feasible solutions with the largest objective function value is equivalent to finding the subset of entries of $\mathbf{c}$ of size at most k that sum to the largest value. We consider the k largest positive entries in $\mathbf{c}$, or fewer if there are not k positive entries, and set the corresponding coordinates in a vector $\tilde{\mathbf{x}}$ to 1 and the remaining entries in $\tilde{\mathbf{x}}$ to 0. Then clearly $\tilde{\mathbf{x}}$ is an optimum basic feasible solution.