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Algorithms, Probability, and Computing **Solutions KW45** **HS18**

Solution 1

As in the lecture notes we use slack variables to get system (1) from the exercise description into form (S1):

$$\begin{array}{lrcl} \text{maximize} & c^T(x_1 - x_2) & & \\ \text{subject to} & A(x_1 - x_2) & \leq & b \\ & x_1, x_2 & \geq & 0 \end{array} \tag{S1}$$

This is the same as

$$\begin{array}{lrcl} \text{maximize} & \begin{pmatrix} c \\ -c \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & & \\ \text{subject to} & (A, -A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \leq & b \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \geq & 0. \end{array} \tag{S2}$$

Using the definition of the dual we get that the dual of (S2) is

$$\begin{array}{lrcl} \text{minimize} & b^T y & & \\ \text{subject to} & (A, -A)^T y & \geq & \begin{pmatrix} c \\ -c \end{pmatrix} \\ & y & \geq & 0. \end{array} \tag{S3}$$

The claim follows since

$$(A, -A)^T y \geq \begin{pmatrix} c \\ -c \end{pmatrix}$$

is equivalent to $A^T y = c$.

Solution 2

1. Neither (P) nor (D) has a feasible solution: An example for this first case can be seen directly below. We observe that neither linear program is feasible because in (P) the constraint $x_1 \leq -1$ contradicts non-negativity of x_1, and in (D) the constraint $-y_2 \geq 1$ contradicts non-negativity of y_2.

$$\begin{array}{rl} \text{(P):} \quad \text{Maximize} & x_1 + x_2 \\ \text{subject to} & x_1, x_2 \geq 0 \\ & x_1 \leq -1 \\ & -x_2 \leq -1 \end{array} \qquad \begin{array}{rl} \text{(D):} \quad \text{Minimize} & -y_1 - y_2 \\ \text{subject to} & y_1, y_2 \geq 0 \\ & y_1 \geq 1 \\ & -y_2 \geq 1 \end{array}$$

2. (P) is unbounded and (D) has no feasible solution: An example for this second case can again be seen below. We observe that (P) is indeed unbounded because we can put $x_1 := 1$ and, at the same time, make x_2 arbitrarily large, which makes also the objective function arbitrarily large. On the other hand, (D) is infeasible because the constraint $-y_2 \geq 1$ directly contradicts non-negativity of y_2.

$$\begin{array}{rl} \text{(P):} \quad \text{Maximize} & x_1 + x_2 \\ \text{subject to} & x_1, x_2 \geq 0 \\ & x_1 \leq 1 \\ & x_1 - x_2 \leq -1 \end{array} \qquad \begin{array}{rl} \text{(D):} \quad \text{Minimize} & y_1 - y_2 \\ \text{subject to} & y_1, y_2 \geq 0 \\ & y_1 + y_2 \geq 1 \\ & -y_2 \geq 1 \end{array}$$

3. (P) has no feasible solution and (D) is unbounded: For this case we simply reverse the roles of the two linear programs from the previous case. Note that in the process we change the names of the variables and we multiply the constraints and objective functions by -1 in order to stay true to the schema from the lecture notes.

$$\begin{array}{rl} \text{(P):} \quad \text{Maximize} & -x_1 + x_2 \\ \text{subject to} & x_1, x_2 \geq 0 \\ & -x_1 - x_2 \leq -1 \\ & x_2 \leq -1 \end{array} \qquad \begin{array}{rl} \text{(D):} \quad \text{Minimize} & -y_1 - y_2 \\ \text{subject to} & y_1, y_2 \geq 0 \\ & -y_1 \geq -1 \\ & -y_1 + y_2 \geq 1 \end{array}$$

4. Both (P) and (D) have a feasible solution: Depicted below is a linear program (P) and its dual (D). On one hand, $\mathbf{x}^* = (1, 2)$ is a feasible solution of (P) with objective value $1 + 2 = 3$. On the other hand, $\mathbf{y}^* = (0.5, 0.5)$ is a feasible solution of (D) with objective value $4 \cdot 0.5 + 2 \cdot 0.5 = 3$. So, clearly, both (P) and (D) are feasible. Additionally, since the objective values of $\mathbf{x}^*$ and $\mathbf{y}^*$ are identical, weak duality tells us that $\mathbf{x}^*$ and $\mathbf{y}^*$ must in fact be optimal solutions of the respective linear programs.

$$\begin{array}{rl} \text{(P):} \quad \text{Maximize} & x_1 + x_2 \\ \text{subject to} & x_1, x_2 \geq 0 \\ & 2x_1 + x_2 \leq 4 \\ & x_2 \leq 2 \end{array} \qquad \begin{array}{rl} \text{(D):} \quad \text{Minimize} & 4y_1 + 2y_2 \\ \text{subject to} & y_1, y_2 \geq 0 \\ & 2y_1 \geq 1 \\ & y_1 + y_2 \geq 1 \end{array}$$

Solution 3

Suppose we are given a linear program L that asks us to maximize the objective function $\mathbf{c}^T\mathbf{x}$ where $\mathbf{c} \neq \mathbf{0}$. Minimization problems can be dealt with analogously, and problems with $\mathbf{c} = \mathbf{0}$ can be solved with one single call to an algorithm that solves the feasibility problem (because any feasible solution is also an optimal solution).

As we will see, with binary search we will not manage to find an optimal solution, but we can get arbitrarily close. So, suppose $\mathbf{x}^*$ is an optimal solution to the given linear program L with objective value $OPT := \mathbf{c}^T\mathbf{x}^*$. Our goal is to find an approximate solution $\tilde{\mathbf{x}}$ that satisfies $OPT - \mathbf{c}^T\tilde{\mathbf{x}} \leq \epsilon$, where $\epsilon > 0$ is an arbitrary but fixed error term.

In order to perform binary search, we need to initialize and maintain upper and lower bounds on the optimum objective value OPT. For this we will need a stronger version of Theorem 4.2, which says that there exists an optimal solution, let us call it $\mathbf{x}^*$ without loss of generality, that is contained in the cube $[-K, K]^n$ with $K \leq 2^{O(\langle L \rangle)}$. We note that the proof in the lecture notes already implies this stronger statement.

It is an easy task to find the vertex $\mathbf{x}_{max}$ (resp., $\mathbf{x}_{min}$) of the cube $[-K, +K]^n$ which maximizes $\mathbf{c}^T\mathbf{x}_{max}$ (resp., minimizes $\mathbf{c}^T\mathbf{x}_{min}$). Indeed, the sign of any coordinate of $\mathbf{c}$ corresponds to the sign of the corresponding coordinate of $\mathbf{x}_{max}$. Also, clearly, $\mathbf{x}_{min} = -\mathbf{x}_{max}$. Since, as we said earlier, $\mathbf{x}^*$ is contained in $[-K, K]^n$ we get that $\alpha := \mathbf{c}^T\mathbf{x}_{max}$ and $\beta := \mathbf{c}^T\mathbf{x}_{min}$ are upper and lower bounds, respectively, for OPT.

Now we can perform binary search for OPT. That is, we let $\gamma := \frac{1}{2}(\alpha + \beta)$ and we add the constraint $\mathbf{c}^T\mathbf{x} \geq \gamma$ to L. We check whether the new program is still feasible. If it is, then we update the lower bound $\beta := \gamma$. If it is not, then we remove the new constraint again and we update the upper bound $\alpha := \gamma$. In any case, the size of the interval $[\beta, \alpha]$ that contains OPT halves in every step of the search. Therefore, the number of steps until we reach $\alpha - \beta \leq \epsilon$ (and therefore also our goal $OPT - \beta \leq \epsilon$) is at most

$$\log_2 \frac{\mathbf{c}^T\mathbf{x}_{max} - \mathbf{c}^T\mathbf{x}_{min}}{\epsilon} = \log_2(2^{O(\langle L \rangle)}) - \log_2(\epsilon) = O(\langle L \rangle),$$

for any fixed $\epsilon > 0$.

Solution 4

We recall from linear algebra that a matrix $R \in \mathbf{R}^{n \times n}$ is a rotation matrix if and only if $R^T = R^{-1}$ (in other words, the columns of R form an orthonormal basis of $\mathbf{R}^n$) and if $\det R = 1$ (if $\det R = -1$ then what we have instead is a rotation combined with a reflection). So, let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{R}^n$ be any set of normalized and pairwise orthogonal vectors with $\mathbf{v}_1 = \mathbf{v}$. Then, $R := (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathbf{R}^{n \times n}$ is a rotation matrix which obviously satisfies $Re_1 = \mathbf{v}_1 = \mathbf{v}$, provided that $\det R = 1$. If $\det R = -1$ then we simply replace the vector $\mathbf{v}_n$, say, with $-\mathbf{v}_n$.