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Algorithms, Probability, and Computing *Fall 2011*
Mid-Term Exam

Candidate:

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions:

1. You can solve the 5 exercises in any order. **You should not be worried if you cannot solve all the exercises!** Not all points are necessary in order to get the best grade. Usually, it pays off to solve fewer tasks but these cleanly. Select wisely, read all tasks carefully first. They are not ordered by difficulty or in any other meaningful way.
2. Check your exam documents for completeness (2 cover pages and 2 pages containing 5 exercises).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. No points will be awarded for unfounded or incomprehensible solutions (except in the multiple-choice parts). You can write your solution in English or German.**
9. You may use anything that has been introduced and proved in the lecture without reproving it. However, if you need something *different* than what we have in the notes, you must write a new proof or at least list all necessary changes.
10. Make sure to write your student-ID (**Legi-number**) on **all** the sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(25)	
2	(25)	
3	(25)	
4	(25)	
5	(25)	
Σ	(125)	

Exercise 1 - Multiple Choice

Consider the following 5 claims and mark the corresponding boxes. Grading: 2 points for a correct marking without a correct justification, 5 points for a correct marking with a correct short justification, and -2 points for a wrongly marked box (you will receive non-negative total points for the exercise in any case).

(a) For a random permutation π on the keys $\{1..n\}$, let T_π be the search tree that we obtain from inserting all n keys, one after the other, in the order given by π. Then T_π has the usual search tree distribution *if and only if* π is distributed uniformly.

[ ] True [ ] False

Justification: ....

....

(b) Consider the process of inserting the keys $\{1, 2, \ldots, n\}$ into an empty treap in the order $\langle 1, 2, \ldots, n \rangle$. The probability that in this process, the rank of the root changes in every step is $\frac{1}{n!}$.

[ ] False [ ] True

Justification: ....

....

(c) Any planar graph with $n > 2$ vertices and $3n - 6$ edges admits at least 2^{n-1} Pfaffian orientations.

[ ] True [ ] False

Justification: ....

....

(d) Suppose that a sequence $\{x_n\}_{n\geq 1}$ with $x_1 > 0$ satisfies $x_n = \sqrt{n} + 2\sum_{i=1}^{n-1} x_i$ for $n \geq 2$. Then $x_n = \mathcal{O}(2^n)$.

[ ] False [ ] True

Justification: ....

....

(e) Suppose we substitute values $x, y, z \in \{0..28\}$ chosen independently and uniformly at random into the polynomial $x^3yz + y^2z^3 + 19 \pmod{29}$. Then the probability of getting zero is less than $\frac{1}{5}$.

[ ] True [ ] False

Justification: ....

....

Exercise 2 - Preprocessing a Matrix

Prove that an $n \times n$ matrix M with real entries can be preprocessed in such a way that queries of the sort "given an interval $[a, b]$ with $a, b \in \mathbf{R}$, report the indices of all rows containing a number from that interval" can be answered in time $\mathcal{O}(n)$ (i.e. describe a suitable data structure and an algorithm for answering the queries along with an analysis of its running time).

HINT: Start with a data structure considered in the lecture, but be careful, you may want to enhance it.

Exercise 3 - Proportional Random Decay

Consider the following random process. We start with the set $\{1..n\}$ of keys. In the first step, we select a key $K_1 \in \{1..n\}$ at random (according to the distribution given below), then we select a smaller key $K_2 \in \{1..K_1 - 1\}$ and then yet a smaller key $K_3 \in \{1..K_2 - 1\}$ and so forth until in the N-th step, $K_N = 1$ is the only choice.

As for the distribution, in the i-th step, once K_{i-1} was drawn, each value $j \in \{1..K_{i-1} - 1\}$ is chosen with a probability proportional to its own value, i.e.

$$\Pr[K_i = j] = \frac{j}{\sum_{l=1}^{K_{i-1}-1} l},$$

where we define $K_0 := n + 1$ for convenience.

Determine $\mathbf{E}[K_1 + K_2 + \ldots + K_N]$.

Exercise 4 - Avoiding Monochromatic 4-Cycles

Let $G = (V, E)$ be a graph on n vertices of maximum degree 21. Let us call a coloring of the edges of G *admissible* if there is no monochromatic 4-cycle (no 4-cycle where all four edges receive the same color).

(a) How many distinct 4-cycles of G can a fixed edge $e \in E$ be part of?

(b) Prove that G admits an admissible edge-coloring using at most 20 colors.

(c) Sketch a randomized algorithm that finds, in polynomial expected running time, given as input a graph G on n vertices of maximum degree 21, an admissible coloring using at most 20 colors and determine its expected running time as a function of n.

Exercise 5 - Ham-Sandwich Queries

Let A and B be two sets of points in general position in the plane $\mathbf{R}^2$. For simplicity, assume $|A| = |B| = n$ is even. A *ham-sandwich cut* for (A, B) is a line l such that half the points from A lie (strictly) above l and half the points from A lie (strictly) below l and, at the same time, half the points from B lie (strictly) above l and half the points from B lie (strictly) below l.

Prove: The sets A and B can be preprocessed in such a way that when a query value $m \in \mathbf{R}$ arrives, your algorithm can tell in time $\mathcal{O}(\log n)$ whether there exists a ham-sandwich cut for (A, B) with slope m and if such a cut exists, it returns a suitable intercept[1] as well.

[1] for a line l with equation $y = mx + b$, we call m the *slope* and b the *intercept*