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November 6, 2012

# *Algorithms, Probability, and Computing* *Fall 2012*
# *Mid-Term Exam*

**Candidate:**

First name: ..........................................................................

Last name: ..........................................................................

Student ID (Legi) Nr.: ..........................................................................

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I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ..........................................................................

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**General remarks and instructions:**

1. You can solve the 5 exercises in any order. **You should not be worried if you cannot solve all the exercises!** Not all points are necessary in order to get the best grade. Usually, it pays off to solve fewer tasks but these cleanly. Select wisely, read all tasks carefully first. They are not ordered by difficulty or in any other meaningful way.
2. Check your exam documents for completeness (2 cover pages and 2 pages containing 5 exercises).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. No points will be awarded for unfounded or incomprehensible solutions (except in the multiple-choice parts). You can write your solution in English or German.**
9. You may use anything that has been introduced and proved in the lecture without reproving it. However, if you need something *different* than what we have in the notes, you must write a new proof or at least list all necessary changes.
10. Make sure to write your student-ID (**Legi-number**) on **all** the sheets (and your name only on this cover sheet).

Good luck!

| | achieved points (maximum) | reviewer's signature |
|---|---|---|
| 1 | (30) | |
| 2 | (30) | |
| 3 | (15) | |
| 4 | (30) | |
| 5 | (25) | |
| $\Sigma$ | (130) | |

## Exercise 1 - Multiple Choice (30 points)

Consider the following 6 claims and mark the corresponding boxes. Grading: 2 points for a correct marking without a correct justification, 5 points for a correct marking with a correct short justification, and -2 points for a wrongly marked box (you will receive non-negative total points for the exercise in any case).

(a) Let $\pi$ be a permutation drawn uniformly at random from all permutations on $n$ elements. Let for $i \in [n]$ denote by $X_i$ the indicator variable that $i$ comes first in $\pi$ among the elements $\{i, i+1, \ldots, n-1, n\}$.

Then the $X_i$ are mutually independent.

[ ] False [ ] True

Justification: ..........................................................................

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(b) Suppose that a sequence $\{x_n\}_{n\geq 1}$ with $x_1 > 0$ satisfies $x_n = -2\sum_{i=1}^{n-1} x_i$ for $n \geq 2$. Then $x_n \in \Theta(2^n)$.

[ ] True [ ] False

Justification: ..........................................................................

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(c) Let $G$ be a simple graph, where multiple edges are allowed. If $G$ has *two* distinct minimum cuts, then for any edge $e \in E(G)$, we have $\mu(G/e) = \mu(G)$.

[ ] False [ ] True

Justification: ..........................................................................

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(d) Let $p(x, y)$ be a polynomial in two real variables with degree $\leq d$. If $p$ is not identically zero, then $q(x) := p(x, x)$ has at most $d$ zeros.

[ ] True [ ] False

Justification: ..........................................................................

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(e) To find the smallest element of a set of $n$ numbers, quickselect takes $\Theta(n^2)$ time in the worst case.

[ ] True [ ] False

Justification: ..........................................................................

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(f) For *any* graph $G$ with at least one edge the number of Pfaffian orientations is even.

[ ] False [ ] True

Justification: ..........................................................................

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## Exercise 2 - Unit Balls (30 points)

Let $t \in \mathbf{N}$. Prove that there exists $k = \Theta(\log t)$ such that the following holds: Any arrangement $A$ of unit balls in $\mathbf{R}^3$ where every point is covered by at most $t$ balls has the following property: There exists a 2-coloring of the balls such that every point that is covered by at least $k$ balls is covered by at least one ball of each color.

You may assume that a unit ball has volume $\frac{4}{3}\pi$ and that every arrangement of $n$ unit balls in $\mathbf{R}^3$ has at most $n^3$ cells.

## Exercise 3 - Checking Matrix Inverse (15 points)

Suppose you are given matrices $A, B \in \mathbf{R}^{n \times n}$ where $B$ is invertible. Give an algorithm that checks whether $A = B^{-1}$ in time $O(n^2)$, returning the right answer with probability at least $\frac{9}{10}$.

## Exercise 4 - Random Search Trees (30 points)

Let $n \in \mathbf{N}$. In this exercise we consider a random binary search tree $T$ on $n$ elements (according to the usual distribution from the lecture).

Let $k_1, k_2, \ldots k_i$ denote the ranks of the keys on the *right* spine of $T$, i.e. on the path from the root to the *largest* key. Determine $E[k_1 + k_2 + \cdots + k_i]$. Note that $i$ is a random number, $k_1$ is the rank of the root of $T$, and $k_i = n$.

## Exercise 5 - Points in the Plane (25 points)

Consider a set of $n$ points $P$ in the plane where no three points are on a line and no points have the same $x$-coordinate. For two points $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$, we consider the quantity $s_{ij} := (y_i - y_j)/(x_i - x_j)$. Describe an algorithm that finds a pair of points that maximizes $s_{ij}$, running in time $O(n \log n)$.