

Solution 1

- (a) The standard BASICMINCUT algorithm can be implemented using an array containing all the degrees of the vertices currently in the graph — for the purpose of efficiently selecting an edge u.a.r. for contraction. The only adaptation needed now is that after each contraction, we scan this array and maintain a global minimal degree ever seen. It is clear that this can be done in linear time per step and thus in $\mathcal{O}(n^2)$ time in total.
- (b) We know from the lecture that contractions can only increase the size of a minimum cut but never decrease it. Since the edges incident to any one vertex always *form* a cut, each of the numbers that we could report in this algorithm corresponds to some cut in the original graph, which readily implies the claim.
- (c) For $n \leq 2$, the claim is empty. Let us now look at $n > 2$, let us fix a graph G of size n and a cut C of size $\mu(G)$. What is the probability that the event we are looking for occurs? There are two cases. Either, G contains a vertex of degree less than $(1 + \alpha)\mu(G)$. In that case, no matter what the further recursion would yield, we will always return a number at most that degree and thus the probability is 1. Or, all vertices in G have degree at least $(1 + \alpha)\mu(G)$. Then, there are at least $(1 + \alpha)\mu(G) \cdot \frac{n}{2}$ edges in the graph and the probability that we contract one from C is thus bounded by $\frac{2}{(1 + \alpha)n}$. But then with the complement of this probability, the new graph G/e will still have a cut of size $\mu(G)$ and by induction, the claim follows.

As for the calculation (which was not required), since $p_\alpha(2) = 1$, we just compute (for

n sufficiently large)

$$\begin{aligned}
p_\alpha(n) &\geq \prod_{i=3}^n \left(1 - \frac{2}{(1+\alpha)i}\right) \\
&= \exp\left(\sum_{i=3}^n \ln\left(1 - \frac{2}{(1+\alpha)i}\right)\right) \\
&\stackrel{(1)}{\geq} \exp\left(\sum_{i=3}^n \left(-\frac{2}{(1+\alpha)i} - \left(\frac{2}{(1+\alpha)i}\right)^2\right)\right) \\
&\stackrel{(2)}{\geq} \exp\left(-\frac{2}{1+\alpha} \left(H_n - \frac{3}{2}\right) - \frac{2\pi^2}{6}\right) \\
&\stackrel{(3)}{\geq} \exp\left(-\frac{2}{1+\alpha} \ln n - \frac{2\pi^2}{6}\right) \\
&= \Omega(n^{\frac{2}{1+\alpha}}),
\end{aligned}$$

where (1) uses the inequality $1 - x \geq e^{-x-x^2}$ for $x \leq 0.68$ and $\frac{2}{(1+\alpha)i} < 2/3 < 0.68$ for all $\alpha > 0$ and $i \geq 3$; where (2) uses that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ and $\frac{2}{1+\alpha} < 2$ for $\alpha > 0$; and where (3) uses $H_n - 3/2 < \ln n$ (the difference between $\ln n$ and H_n approaches the Euler-Mascheroni constant γ , which is about 0.577).

Solution 2

Given a graph $G = (V, E)$, let N be the number of minimum cuts in G . We want to show that $N \leq \binom{n}{2}$ where $n := |V|$.

Let C_1, \dots, C_N be the minimum cuts in G . Then, we know that for each $i \in \{1..N\}$

$$\Pr[C_i \text{ is found by Karger's algorithm BASICMINCUT}(G)] \geq \frac{1}{\binom{n}{2}}.$$

Now, we observe that for each two distinct indices $i, j \in \{1..N\}$ the events “ C_i is found by BASICMINCUT(G)” and “ C_j is found by BASICMINCUT(G)” are disjoint (i.e. they never happen at the same time). To see this, consider the graph obtained at the termination of BASICMINCUT(G). It has only two vertices and these vertices (together with the “contraction history”) uniquely determine a partition of the vertex set V . So we cannot get two different minimum cuts from one execution of the algorithm. Therefore, it follows that

$$\begin{aligned}
&\Pr[\text{a minimum cut is found by BASICMINCUT}(G)] \\
&= \sum_{i=1}^N \Pr[C_i \text{ is found by BASICMINCUT}(G)] \geq \frac{N}{\binom{n}{2}}.
\end{aligned}$$

Since $\Pr[\text{a minimum cut is found by BASICMINCUT}(G)] \leq 1$ (because it is a probability!), we obtain $N \leq \binom{n}{2}$.

Solution 3

- (a) Observe that there are three (potentially empty) sets of edges $e = \{u, v\}$ that are important in this scenario:

$$\begin{aligned} E_1 &:= \{e = \{u, v\} \mid u \in A \cap B \text{ and } v \in V \setminus (A \cup B)\} \\ E_2 &:= \{e = \{u, v\} \mid u \in (A \setminus B) \cup (B \setminus A) \text{ and } v \in V \setminus (A \cup B)\} \\ E_3 &:= \{e = \{u, v\} \mid u \in A \cap B \text{ and } v \in (A \setminus B) \cup (B \setminus A)\}. \end{aligned}$$

It is easy to see that $f(A \cap B) + f(A \cup B) = |E_1| + |E_3| + |E_1| + |E_2|$, while $f(A) + f(B) \geq 2|E_1| + |E_2| + |E_3|$, the last inequality holding because some edges between $A \setminus B$ and $B \setminus A$ may appear both in $f(A)$ and $f(B)$.

- (b) Let k be the size of a minimum cut. Then by (a) we get $f(A \cap B) + f(A \cup B) \leq f(A) + f(B) = k + k$, which together with $f(C) \geq k$ for any set $C \neq \emptyset, V$, implies $f(A \cap B) = f(A \cup B) = k$.
- (c) Suppose towards a contradiction that $S \neq S'$, with $S, S' \subset V$ and $s \in S \cap S'$ are such that $C(S)$ and $C(S')$ are both minimum cuts and $|S| = |S'|$ is minimal. Note that as $t \notin S \cup S'$ because they are both cuts, we must have $S \cup S' \neq V$, and so we can in a similar way to part (b) prove that $S \cap S'$ is a minimum cut with $|S \cap S'| < |S| = |S'|$, a contradiction.

Solution 4

- (a) Since G is connected, there are at least $n - 1$ edges. If there are at least n edges, then $\Pr[\mu(G) \neq \mu(G/e)] \leq \frac{1}{n}$ since in the worst case the minimum cut is unique and the probability of contracting a given edge is at most the claimed bound.

If there are only $n - 1$ edges the graph is a tree. Then contracting any edge keeps the graph a tree with 1 fewer vertex which means that $\Pr[\mu(G) \neq \mu(G/e)] = 0 \leq \frac{1}{n}$.

- (b) BASICMINCUT will succeed if it never contracts a given edge in the cut of size 1. This happens with probability at least

$$\begin{aligned} &\left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n-1}\right) \cdots \left(1 - \frac{1}{3}\right) \\ &= \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n-1}\right) \cdots \left(\frac{2}{3}\right) = \frac{2}{n}. \end{aligned}$$

- (c) A contraction of an edge changes the degree of at most 2 vertices. Therefore there is still at least one vertex of degree $k \Rightarrow$ minimum cut is still of size k .

(d) If there are 3 or more vertices of degree k we are done by (c). If there are 0 or 1 vertices of degree k , then the number of edges is at least $((n-1)(k+1)+k)/2 = (n(k+1)-1)/2$ so that when we fix a minimum cut of size k , the probability of contracting one of the edges of the minimum cut is at most

$$\frac{k}{(n(k+1)-1)/2} = \frac{2k}{n(k+1)-1}.$$

It remains to consider the case that there are two vertices of degree k . If these two vertices are not adjacent, then there are two disjoint minimum cuts which means that $\Pr[\mu(G) \neq \mu(G/e)] = 0$. If they are adjacent, then the minimum cut may only change if we contract one of the at most $k-1$ edges between the two vertices (they can't be connected via k edges since the graph is connected and $n \geq 3$). Since there are at least $(2k + (n-2)(k+1))/2$ edges, the failure probability is at most

$$\frac{2(k-1)}{2k + (n-2)(k+1)} \leq \frac{2k-2}{n(k+1)-2} \leq \frac{2k-1}{n(k+1)-1} \leq \frac{2k}{2n(k+1)-1}.$$

The second to last inequality holds because $a := 2k-1 \leq n(k+1)-1 =: b$ and then

$$\frac{a-1}{b-1} \leq \frac{a}{b} \Leftrightarrow -b \leq -a \Leftrightarrow a \leq b.$$