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Algorithms, Probability, and Computing **Solutions KW46** **HS19**

Solution 1

First, we check with one call to the oracle whether the given system of linear inequalities has a solution. If the answer is `NO` then we stop and also output `NO`. If the answer is `YES` then we proceed as follows.

(a) If there are only equations in the system, then this just means that we have a system $A\mathbf{x} = \mathbf{b}$ of linear equations, for which we can find a solution in polynomial time by Gauss elimination.[1] So, in this case we need no additional calls to the oracle.

(b) If there is at least one inequality, say $\mathbf{a}\mathbf{x} \le b$, then we replace it by $\mathbf{a}\mathbf{x} = b$. If the new, more constrained, system still has a solution (which can be checked with one additional call to the oracle), then we can recursively find a solution to the original system by finding a solution to the more constrained system. If the new, more constrained, system turns out to have no solution, then we drop the constraint $\mathbf{a}\mathbf{x} \le b$ completely to obtain a smaller system, which again can be solved recursively. This is a sound strategy because replacing a constraint $\mathbf{a}\mathbf{x} \le b$ with $\mathbf{a}\mathbf{x} = b$ can turn a feasible problem into an infeasible one if and only if the hyperplane corresponding to $\mathbf{a}\mathbf{x} \le b$ is not part of the boundary of the feasible region (in other words, it is redundant).

Since we need one initial call to the oracle, and exactly one call per inequality that we get rid of (either by replacing it with an equality or by dropping it completely), the total number of calls to the oracle will be $m + 1$, where m is the number of inequalities in the original system.

Solution 2

(i) "$\Rightarrow$": Assume that G is connected, and let $S \subseteq V$, $\emptyset \neq S \neq V$. Then $(S, V \setminus S)$ is a partition of V into two nonempty subsets. The set $\delta(S)$ consists exactly of those edges that have one endpoint in S and the other one in $V \setminus S$. If this set were empty then G would be disconnected (because there could be no path from any vertex in S to any vertex in $V \setminus S$).

[1] When the computation has to be done exactly, naive implementations of Gauss elimination can lead to an exponential blow-up of the encoding size of intermediate results. However, there are more clever implementations which do not have this problem and which do run in polynomial time.

"$\Leftarrow$": Let us write "condition $(*)$" for the condition $\delta(S) \neq \emptyset$ for all $S \subseteq V$, $\emptyset \neq S \neq V$. Assume that G satisfies condition $(*)$. Let $v \in V$. We want to show that there is a path from v to every other vertex. To this end let C_v denote the connected component of v in G. Applying condition $(*)$ to C_v, there are only three possibilities: (1) There is an edge from C_v to $V \setminus C_v$; but this contradicts the definition of a connected component. (2) $C_v = \emptyset$; but this contradicts $v \in C_v$. (3) $C_v = V$, q.e.d.

(ii) Let G be the Petersen graph. It is known that G is non-Hamiltonian, so that the Subtour LP cannot have an integer solution. It is also known (or obvious) that G is 3-connected as well as 3-regular. From this it follows that setting $c_e = 2/3$ for all edges e gives a feasible solution to the Subtour LP.

(If you have not seen the Petersen graph before, a websearch will give you many pictures of it. It happens to be one of the smallest non-Hamiltonian graphs out there, which should explain why we use it for this question.)

(iii) Let x be a feasible point and assume that there is $\eta \in E$ with $x_\eta > 1$. Write $\eta = \{u, v\}$ (the two endpoints of the bad edge). The first contraint of our LP reads $\sum_{e\in\delta(v)} x_e = \sum_{e\in\delta(u)} x_e = 2$. In other words, $\sum_{e\in\delta(\{u,v\})} x_e = 4 - 2x_\eta < 2$, a contradiction to the second constraint applied to $S := \{u, v\}$. (In the last step we have used the assumption $|V| > 3$, which guarantees $S \neq V$.)

Solution 3

Let x be a feasible point of the Tight Spanning Tree LP. Let $S \subseteq V$, $\emptyset \neq S \neq V$. We want to show that x also satisfies the constraint

$$\sum_{e\in\delta(S)} x_e \geq 1. \qquad (*)$$

We know

$$\sum_{e\in E\cap\binom{S}{2}} x_e \leq |S| - 1,$$

$$\sum_{e\in E\cap\binom{V\setminus S}{2}} x_e \leq |V \setminus S| - 1,$$

which together is at most $|V| - 2 = n - 2$. Due to the first constraint, $\sum_{e\in E} x_e = n - 1$, the values x_e of the remaining edges (those not in $\binom{S}{2}$ or $\binom{V\setminus S}{2}$) must sum up to at least 1. This is exactly the statement $(*)$ that we wanted to show.

Solution 4

Let $S = \{s_1, \ldots, s_n\}$, and let the numbering be chosen in such a way that $\min_{x \in S} c^T x = c^T s_1$. Let $y \in \text{conv}(S)$. By definition of the convex hull there is $\lambda \in \mathbf{R}^n$ such that $\lambda \geq 0$, $\mathbf{1}_n^T \lambda = 1$ and $y = \sum_{i=1}^n \lambda_i s_i$. We find

$$c^T y = \sum_{i=1}^n \lambda_i c^T s_i \geq \sum_{i=1}^n \lambda_i c^T s_1 = \left(\min_{x \in S} c^T x \right) \sum_{i=1}^n \lambda_i = \min_{x \in S} c^T x.$$

Since $y \in \text{conv}(S)$ was arbitrary, we obtain

$$\min_{y \in \text{conv}(S)} c^T y \geq \min_{x \in S} c^T x$$

and "$\leq$" clearly holds anyways.

Solution 5

Consider the edge set $E' \subseteq E$ whose characteristic vector corresponds to some feasible $x \in \{0, 1\}^E$. Recall that the constraints

$$\sum_{e \in E \cap \binom{S}{2}} x_e \leq |S| - 1 \ , \quad \text{for all } S \subseteq V, \ \emptyset \neq S \neq V$$

imply that the subgraphs $(S, E' \cap S)$ for $\emptyset \neq S \neq V$ are acyclic. But because x also satisfies the constraint

$$\sum_{e \in E} x_e = n$$

we know that the graph $G' = (V, E')$ has a cycle. Therefore this cycle has to be a Hamilton cycle and G' can not contain other edges since a Hamilton cycle has n edges. Every characteristic vector of a Hamilton cycle satisfies all the constraints so the characteristic vectors are exactly the Hamilton cycles in G.