ETH

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Algorithms, Probability, and Computing **Solutions for SPA 2** **HS19**

This is a draft version. After we have graded your assignments, we will publish an improved version based on the feedback that we have obtained from what you have written.

Solution 1

(a) The left-hand side of the expression can be written as

$$\mathbb{E}\left[y_1^{r_1}\cdots y_q^{r_q}\right] \tag{1}$$

where $y_1, \ldots, y_q$ are independent uniform, random variables with values -1 or 1, and $r_1 + \ldots + r_q = 2k+1$. Therefore, $j \in \{1, \ldots, q\}$ exists such that r_j is odd and thus $\mathbb{E}[y_j^{r_j}] = 0$. Hence Equation 1 becomes:

$$\mathbb{E}\left[y_1^{r_1}\cdots y_q^{r_q}\right] = \mathbb{E}\left[y_1^{r_1}\cdots y_{j-1}^{r_{j-1}}\right]\cdot\mathbb{E}\left[y_j^{r_j}\right]\cdot\mathbb{E}\left[y_{j+1}^{r_{j+1}}\cdots y_q^{r_q}\right] = 0.$$

(b) Let $m = 2k+1$ and define $Z \subseteq S_n$ as $Z = \{\pi \in S_n \mid a_{1,\pi(1)}\cdots a_{n,\pi(n)} = 1\}$. Then

$$\begin{aligned}\mathbb{E}\left[\left(\det(B)\right)^m\right] &= \mathbb{E}\left[\left(\sum_{\pi\in Z}\text{sign}(\pi)\epsilon_{1,\pi(1)}\cdots\epsilon_{n,\pi(n)}\right)^m\right]\\ &= \sum_{\pi_1,\ldots,\pi_m\in Z}\prod_{1\le i\le n}\Big(\text{sign}(\pi_i)\cdot\mathbb{E}\left[\epsilon_{i,\pi_1(i)}\cdots\epsilon_{i,\pi_m(i)}\right]\Big).\end{aligned}$$

Hence, the result from the task (a) gives us that each of the expectations above are zero, completing the proof.

Solution 2

(a) We define the parameters a_{ijk} as in the hint.

Note that for all $(i,j) \in P$, if $i = j$, we can trivially assign the flow from i to j such that it does not pass through any arc. Whether this is clockwise or anticlockwise is by convention. We will only then solve the problem for the remaining pairs. In other words, we can assume that for all $(i,j) \in P, i \neq j$.

Consider the following integer program ("IP")

$$\begin{array}{lll}\text{minimise} & w & \\ \text{subject to} & \displaystyle\sum_{(i,j)\in P}\Big(x_{ij}a_{ijk} + (1-x_{ij})(1-a_{ijk})\Big) \le w & \forall k \in [n]\\ & x_{ij} \in \{0,1\} & \forall (i,j) \in P\end{array}$$

Note that the mapping of x to the characteristic vector of the set $\{(i,j) \in P \mid$ the flow from i to j is counterclockwise$\}$ is a bijection. Further, for each $k \in [n]$, $\sum_{(i,j)\in P} x_{ij} a_{ijk}$ corresponds to the sum of anticlockwise flows through arc k, while $\sum_{(i,j)\in P}(1-x_{ij})(1-a_{ijk})$ the sum of clockwise flows. Hence, the left hand side of the first constraint of IP corresponds to w_i. As such, given x, the minimum value of w such that the IP stays feasible then corresponds to $\max_i w_i$ or the circle width of the corresponding flow assignment of the problem. Therefore, a solution (x, w) that is optimal to IP has a one-to-one correspondence with an assignment of flows that achieve the optimal circle width for the problem. It follows that the IP models the problem.

(b) Consider the linear problem (L) which is identical to IP, except that the last constraint is replaced by $0 \le x_{ij} \le 1$, $\forall (i,j) \in P$.

Suppose x^* and w^* form an optimal solution of (L). As a feasible solution of the IP is also a fesaible solution of (L), $w^* \le \text{OPT}$.

For $(i,j) \in P$, let $\overline{x}_{ij} := [x^*_{ij}]$, i.e., the value of x^*_{ij} rounded to the nearest integer. Let $\overline{w} := \max_k \overline{w}_k$, where $\overline{w}_i := \sum_{(i,j)\in P}\Big(\overline{x}_{ij} a_{ijk} + (1-\overline{x}_{ij})(1-a_{ijk})\Big)$. Following the argument in part (a), $\overline{x}$ corresponds to an assignment of flows for the original problem with the resulting circle width $\overline{w}$.

For $a \in [0,1]$, checking both cases of $a \in [0, 0.5)$ and $a \in [0.5, 1]$, we have $[a] \le 2a$. Applying this statement, we have for all $k \in [n]$,

$$
\begin{aligned}
\overline{w}_k = \sum_{(i,j)\in P}\Big(\overline{x}_{ij} a_{ijk} + (1-\overline{x}_{ij})(1-a_{ijk})\Big) &= \sum_{(i,j)\in P}\Big([x^*_{ij}] a_{ijk} + (1-[x^*_{ij}])(1-a_{ijk})\Big) \\
&= \sum_{(i,j)\in P}\Big([x^*_{ij}] a_{ijk} + [1-x^*_{ij}](1-a_{ijk})\Big) \\
&\le \sum_{(i,j)\in P}\Big(2x^*_{ij} a_{ijk} + 2(1-x^*_{ij})(1-a_{ijk})\Big) \\
&\le 2w^* \le 2\text{OPT}
\end{aligned}
$$

As $\overline{w} = \max_k \overline{w}_k$, this implies $\overline{w} \le 2\text{OPT}$.

Hence, we find a solution of the original problem where the circle width is at most 2OPT.

The algorithm then is as follows:

- Solve (L) to obtain an optimal solution (x^*, w^*).
- For $(i,j) \in P$, if $x_{ij} \ge 0.5$, we assign an anticlockwise flow from i to j. Otherwise, we assign a clockwise flow from i to j.
- Output the assignment.

Solution 3

(a) Let P denote the convex polyhedron $\bigcap_{j=0}^{i} h^*_j$, and let u be a lowest vertex of P. It is clear that the x_d-coordinate of u is higher than or equal to that of v. Since $u \in h^*_i$ and $v \notin h^*_i$, the line segment $\overline{uv}$ must intersect h_i at a point u', and the x_d-coordinate of u' is not higher than u. Moreover, u' lies in a facet f_i of P, and f_i belongs to h_i. If u' is a vertex of P, then the statement is true. So we assume that u' is not a vertex of P. If f_i is bounded, then the

x_d-coordinate of at least one vertex of f_i is not higher than of u'. If f_i is unbounded, due to the existence of h_0, f_i must be lower bounded with respect to the x_d-axis. Moreover, since any d boundary hyperplanes intersects at a point and d hyperplanes have been intersected, f_i own at least one lowest vertex, and this vertex is not higher than of u'. Therefore, one lowest vertex of P lies in h_i.

(b) We first analyze the probability that $v \notin h_i^*$. If $v \notin h_i^*$, the new computed lowest vertex will lie in h_i. Since a vertex is defined by d boundary hyperplanes and each half-space in the first i inserted ones is equally likes to be the i-th one, the corresponding probability is $\frac{d}{i}$.

Then, we prove the statement by induction.

When $d = 2$, the running time is exactly that of Random-2D-LP and is expected $\mathcal{O}(n) = \mathcal{O}(d!n)$.

Assume that for $d = k \geq 2$, the running time is expected $\mathcal{O}(k!n)$.

When $d = k + 1$, since the probability that $v \notin h_i^*$ is $\frac{k+1}{i}$, the expected time for the i-th insertion is $\frac{k+1}{i} \cdot \mathcal{O}\big(k! \cdot i\big) = \mathcal{O}\big((k+1)!\big)$. Since there are $n - d + 1$ insertions, the linearity of expectation implies that the Random-LP takes expected $\mathcal{O}\big((k+1)!n\big)$ time.

(c) We aim at the asymptotically maximum value of d such that $d! \cdot n = \Theta(n^2)$. In this situation, $d! = \Theta(n)$, i.e., $c_1 \cdot n \leq d! \leq c_2 \cdot n$, for some constants c_1 and c_2. Therefore, we have

$$\log c_1 + \log n \leq \log d! \leq \log c_2 + \log n.$$

According to the hint, we have

$$\frac{d}{2} \log \frac{d}{2} \leq \log d! \leq d \log d.$$

If $d = \frac{\log n}{\log \log n}$,

$$d \log d = \frac{\log n}{\log \log n} \log \frac{\log n}{\log \log n} \leq \frac{\log n}{\log \log n} \log \log n = \log n \leq \log c_1 + \log n.$$

If $d = 2\frac{\log n}{\log \log n}$,

$$d \log d = 2\frac{\log n}{\log \log n} \log 2\frac{\log n}{\log \log n} = \frac{\log n}{\log \log n} \cdot 2 \cdot \log(2\frac{\log n}{\log \log n})$$

$$= \frac{\log n}{\log \log n}(2 + 2 \log \log n - 2 \log \log \log n) \geq \frac{\log n}{\log \log n}(2 + \log \log n) \geq \log c_2 + \log n.$$

For $n \geq 4$, $2 \cdot \frac{\log n}{\log \log n} \geq 4$, so that we can choose c_2 as $2^4 = 16$ and c_1 as any constant smaller than c_2. Moreover, for any fixed $n \geq 4$, $c \cdot \frac{\log n}{\log \log n}$ is a continuous function of c and increases with c. Therefore, there exists $d \in [\frac{\log n}{\log \log n}, 2\frac{\log n}{\log \log n}]$ such that $c_1 \cdot n \leq d! \leq c_2 \cdot n$, and $d \in \Theta(\frac{\log n}{\log \log n})$.