Eidgenössische Technische Hochschule Zürich
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Algorithms, Probability, and Computing **Exercises KW41** **HS20**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16

  Group B: Wed 14–16

  Group C: Wed 16–18

  Group D: Wed 16–18

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise class on October 7, 2020. Please hand in your solutions via Moodle, no later than 10 pm October 4.

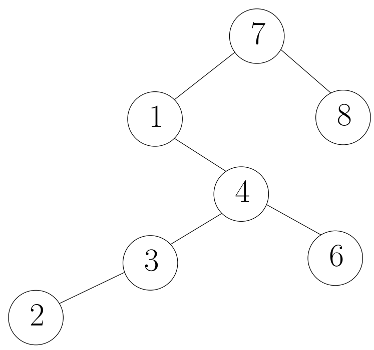
Exercise 1

Let $n \in \mathbf{N}$. Show that the expected number of nodes of depth $n-1$ in a random search tree for n keys is $\frac{2^{n-1}}{n!}$. What is the probability that there is a node of depth $n-1$?

Exercise 2

Let S_n denote the number of keys that are descendants of the smallest key. For example, in the tree below, $S_n = 5$, because the elements $1, 2, 3, 4, 6$ are descendants of 1.

Compute $\mathbf{E}[S_n]$.



Exercise 3

Determine closed forms for the following recursively defined series:

(1) For $n \in \mathbf{N}$,

$$a_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 1 + \frac{1}{n} \sum_{i=1}^{n-1} a_i, & \text{otherwise.} \end{cases}$$

(2) For $n \in \mathbf{N}$,

$$b_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 2 + \sum_{i=1}^{n-1} b_i, & \text{otherwise.} \end{cases}$$

(3) For $n \in \mathbf{N}_0$,

$$c_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ n - 1 + \sum_{i=1}^{n} \frac{c_{i-1} + c_{n-i}}{2}, & \text{otherwise.} \end{cases}$$

(4) For $n \in \mathbf{N}_0$,

$$d_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ 1 + 2 \sum_{i=0}^{n-1} (-1)^{n-i} d_i, & \text{otherwise.} \end{cases}$$

(5) For $n \in \mathbf{N}_0$,

$$e_n = \begin{cases} 1, & \text{if } n = 0, \text{ and} \\ 1 + n\, e_{n-1}, & \text{otherwise.} \end{cases}$$

Exercise 4

Let $n \in \mathbf{N}$. Determine the expected number of leaves in a random search tree for n keys.

Exercise 5

Consider the process of inserting the keys $\{1, 2, \dots, n\}$ into an empty treap in the order $(1, 2, \dots, n)$.

(a) During this process, what is the expected number of changes of the root of the treap? (We also count the very first insertion as a change of the root.)

(b) For a given key i: What is the probability that i occurs as the right child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

(c) What is the expected number of elements that occur as the left child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

Exercise 6

Let $i, j, n \in \mathbf{N}$, $i < j \leq n$. What is the probability that the randomized procedure quicksort() applied to a set of n numbers compares the element of rank i with the element of rank j?
Hint: If you are stuck, you might want to read section 2.4 of the script.