Solution 1

(a) The standard BasicMinCut algorithm can be implemented using an array containing all the degrees of the vertices currently in the graph — for the purpose of efficiently selecting an edge u.a.r. for contraction. The only adaptation needed now is that after each contraction, we scan this array and maintain a global minimal degree ever seen. It is clear that this can be done in linear time per step and thus in $O(n^2)$ time in total.

(b) We know from the lecture that contractions can only increase the size of a minimum cut but never decrease it. Since the edges incident to any one vertex always form a cut, each of the numbers that we could report in this algorithm corresponds to some cut in the original graph, which readily implies the claim.

(c) For $n \leq 2$, the claim is empty. Let us now look at $n > 2$, let us fix a graph $G$ of size $n$ and a cut $C$ of size $\mu(G)$. What is the probability that the event we are looking for occurs? There are two cases. Either, $G$ contains a vertex of degree less than $(1 + \alpha)\mu(G)$. In that case, no matter what the further recursion would yield, we will always return a number at most that degree and thus the probability is 1. Or, all vertices in $G$ have degree at least $(1 + \alpha)\mu(G)$. Then, there are at least $(1 + \alpha)\mu(G) \cdot \frac{n}{2}$ edges in the graph and the probability that we contract one from $C$ is thus bounded by $\frac{2}{(1 + \alpha)n}$. But then with the complement of this probability, the new graph $G/e$ will still have a cut of size $\mu(G)$ and by induction, the claim follows.

As for the calculation (which was not required), since $p_\alpha(2) = 1$, we just compute (for
n sufficiently large)

\[ p_\alpha(n) \geq \prod_{i=3}^{n} \left( 1 - \frac{2}{(1+\alpha)i} \right) \]

\[ = \exp \left( \sum_{i=3}^{n} \ln \left( 1 - \frac{2}{(1+\alpha)i} \right) \right) \]

\[ \geq \exp \left( \sum_{i=3}^{n} \left( -\frac{2}{(1+\alpha)i} - \left( \frac{2}{(1+\alpha)i} \right)^2 \right) \right) \]

\[ \geq \exp \left( -\frac{2}{1+\alpha} \left( H_n - \frac{3}{2} - \frac{2\pi^2}{6} \right) \right) \]

\[ \geq \exp \left( -\frac{2}{1+\alpha} \ln n - \frac{2\pi^2}{6} \right) \]

\[ = \Omega(n^{-1/\alpha}), \]

where (1) uses the inequality \( 1 - x \geq e^{-x-x^2} \) for \( x \leq 0.68 \) and \( \frac{2}{(1+\alpha)i} < 2/3 < 0.68 \) for all \( \alpha > 0 \) and \( i \geq 3 \); where (2) uses that \( \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \) and \( \frac{2}{1+\alpha} < 2 \) for \( \alpha > 0 \); and where (3) uses \( H_n - 3/2 < \ln n \) (the difference between \( \ln n \) and \( H_n \) approaches the Euler-Mascheroni constant \( \gamma \), which is about 0.577).

**Solution 2**

Given a graph \( G = (V, E) \), let \( N \) be the number of minimum cuts in \( G \). We want to show that \( N \leq \binom{n}{2} \) where \( n := |V| \).

Let \( C_1, \ldots, C_N \) be the minimum cuts in \( G \). Then, we know that for each \( i \in \{1..N\} \)

\[ \Pr[C_i \text{ is found by Karger’s algorithm BasicMinCut}(G)] \geq \frac{1}{\binom{n}{2}}. \]

Now, we observe that for each two distinct indices \( i, j \in \{1..N\} \) the events “\( C_i \text{ is found by BasicMinCut}(G) \)” and “\( C_j \text{ is found by BasicMinCut}(G) \)” are disjoint (i.e. they never happen at the same time). To see this, consider the graph obtained at the termination of BasicMinCut(\( G \)). It has only two vertices and these vertices (together with the “contraction history”) uniquely determine a partition of the vertex set \( V \). So we cannot get two different minimum cuts from one execution of the algorithm. Therefore, it follows that

\[ \Pr[\text{a minimum cut is found by BasicMinCut}(G)] = \sum_{i=1}^{N} \Pr[C_i \text{ is found by BasicMinCut}(G)] \geq \frac{N}{\binom{n}{2}}. \]
Since \( \Pr[\text{a minimum cut is found by } \text{BasicMinCut}(G)] \leq 1 \) (because it is a probability!), we obtain \( N \leq \binom{n}{2} \).

**Solution 3**

(a) Observe that there are three (potentially empty) sets of edges \( e = \{u, v\} \) that are important in this scenario:

\[
    \begin{align*}
    E_1 &:= \{ e = \{u, v\} \mid u \in A \cap B \text{ and } v \in V \setminus (A \cup B) \} \\
    E_2 &:= \{ e = \{u, v\} \mid u \in (A \setminus B) \cup (B \setminus A) \text{ and } v \in V \setminus (A \cup B) \} \\
    E_3 &:= \{ e = \{u, v\} \mid u \in A \cap B \text{ and } v \in (A \setminus B) \cup (B \setminus A) \}.
    \end{align*}
\]

It is easy to see that \( f(A \cap B) + f(A \cup B) = |E_1| + |E_2| + |E_3| + |E_4| \), while \( f(A) + f(B) \geq 2|E_1| + |E_2| + |E_3| \), the last inequality holding because some edges between \( A \setminus B \) and \( B \setminus A \) may appear both in \( f(A) \) and \( f(B) \).

(b) Let \( k \) be the size of a minimum cut. Then by (a) we get \( f(A \cap B) + f(A \cup B) \leq f(A) + f(B) = k + k \), which together with \( f(C) \geq k \) for any set \( C \neq \emptyset, V \), implies \( f(A \cap B) = f(A \cup B) = k \).

(c) Suppose towards a contradiction that \( S \neq S' \), with \( S, S' \subseteq V \) and \( s \in S \cap S' \) are such that \( C(S) \) and \( C(S') \) are both minimum cuts and \( |S| = |S'| \) is minimal. Note that as \( t \notin S \cup S' \) because they are both cuts, we must have \( S \cup S' \neq V \), and so we can in a similar way to part (b) prove that \( S \cap S' \) is a minimum cut with \( |S \cap S'| < |S| = |S'| \), a contradiction.

**Solution 4**

(a) Since \( G \) is connected, there are at least \( n - 1 \) edges. If there are at least \( n \) edges, then \( \Pr[\mu(G) \neq \mu(G/e)] \leq \frac{1}{n} \) since in the worst case the minimum cut is unique and the probability of contracting a given edge is at most the claimed bound.

If there are only \( n - 1 \) edges the graph is a tree. Then contracting any edge keeps the graph a tree with 1 fewer vertex which means that \( \Pr[\mu(G) \neq \mu(G/e)] = 0 \leq \frac{1}{n} \).

(b) \text{BasicMinCut} will succeed if it never contracts a given edge in the cut of size 1. This happens with probability at least

\[
\left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n-1}\right) \cdots \left(1 - \frac{1}{3}\right) = \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n-1}\right) \cdots \left(\frac{2}{3}\right) = \frac{2}{n}.
\]

(c) A contraction of an edge changes the degree of at most 2 vertices. Therefore there is still at least one vertex of degree \( k \Rightarrow \) minimum cut is still of size \( k \).
(d) If there are 3 or more vertices of degree $k$ we are done by (c). If there are 0 or 1 vertices of degree $k$, then the number of edges is at least \((n-1)(k+1)+k)/2 = (n(k+1)-1)/2\) so that when we fix a minimum cut of size $k$, the probability of contracting one of the edges of the minimum cut is at most

\[
\frac{k}{(n(k+1)-1)/2} = \frac{2k}{n(k+1)-1}.
\]

It remains to consider the case that there are two vertices of degree $k$. If these two vertices are not adjacent, then there are two disjoint minimum cuts which means that $\Pr[\mu(G) \neq \mu(G/e)] = 0$. If they are adjacent, then the minimum cut may only change if we contract one of the at most $k-1$ edges between the two vertices (they can’t be connected via $k$ edges since the graph is connected and $n \geq 3$). Since there are at least \((2k+(n-2)(k+1))/2\) edges, the failure probability is at most

\[
\frac{2(k-1)}{2k+(n-2)(k+1)} \leq \frac{2k-2}{n(k+1)-2} \leq \frac{2k-1}{n(k+1)-1} \leq \frac{2k}{2n(k+1)-1}.
\]

The second to last inequality holds because $a := 2k-1 \leq n(k+1)-1 =: b$ and then

\[
\frac{a-1}{b-1} \leq \frac{a}{b} \iff -b \leq -a \iff a \leq b.
\]

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