Eidgenössische Technische Hochschule Zürich
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Algorithms, Probability, and Computing — Exercises KW39 — HS21

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16

  Group B: Wed 14–16

  Group C: Wed 16–18

  Group D: Wed 16–18

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on September 29, 2021. You can hand in your solutions during the lecture break on September 27 or September 28 or via Moodle, no later than 4 pm at September 28.

Exercise 1

Let $G = (V, E)$ be a connected graph with weights $w : E \to R$ on the edges, and define the edge boundary of set S to be

$$\partial(S) := \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$$

Assume that for every non-empty vertex set $S \subset V$, the edge with the minimum weight in $\partial(S)$ is unique.

Prove that G has a unique MST. Conclude that if the weight function w is injective (i.e., no two edges have the same weight), G contains exactly one MST.

Exercise 2

You already know that for a connected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, the expected running time of *Randomized Minimum Spanning Tree Algorithm (G)* (see page 5 in the lecture notes) is equal to $\mathcal{O}(m)$.

(i) Prove that the worst-case running time of the algorithm is equal to $\mathcal{O}(\min\{n^2, m \log n\})$.

(ii) Prove that the running time of the algorithm is equal to $\mathcal{O}(m)$ with probability $1 - o(1)$ in the following two steps.

(a) Let $D(n, m)$ be the worst-case running time of the recursive algorithm without considering the two recursive calls, and let $T(n, m)$ be the worst-case running time of the recursive algorithm. It is clear that $\mathbf{D(n, m) = \mathcal{O}(n + m)}$, and by (i), $T(n, m) = \mathcal{O}(\min\{n^2, m \log n\})$. Figure 1 represents a binary tree of running times in which every first recursive call works on a graph with at most $\frac{n}{8}$ vertices and at most $\frac{3}{4}m$ edges, every second recursive call works on a graph with at $\frac{n}{8}$ vertices and at most $\frac{3}{8}n$ edges, and the worst-case function $T(n^{2/5}, \infty)$ will be applied when the number of vertices has shrunk down to $n^{2/5}$.

Prove that there exists a constant $c_3 > 0$ such that the sum of all running times in the nodes of the tree depicted in Figure 1 is bounded from above by $c_3 \cdot (n + m)$.

(b) Prove that for any connected input graph G, the running time of the algorithm is dominated by the sum over all nodes in the tree depicted in Figure 1 with probability $1 - o(1)$ (i.e., a number that tends to 1 as $n \to \infty$).

Hint: Let G_1 *and* G_2 *be the two graphs for the first and second recursive calls, respectively. Call* G_1 bad *if* G_1 *has more than* $\frac{n}{8}$ *vertices or more than* $\frac{3}{4}m$ *edges, and call* G_2 bad *if* G_2 *has more than* $\frac{n}{8}$ *vertices or* $\frac{3}{8}n$ *edges. You might apply the Chernoff bound to bound the probability that* G_1 *or* G_2 *is bad. For the Chernoff bound, see the help sheet on the website of the course.*

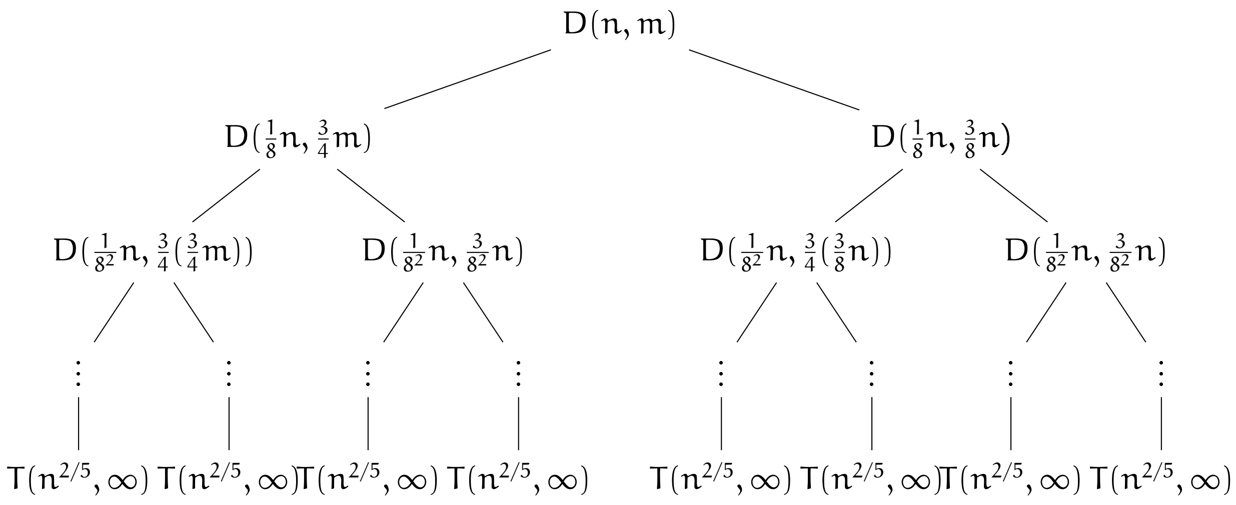
Exercise 3

For a graph $G = (V, E)$, a cut is the partition of the vertex set V into two disjoint sets V_1 and V_2 and the size of the cut is the number of edges between V_1 and V_2.

(i) Assume that $|V|$ is even. We say a cut is balanced if $|V_1| = |V_2| = |V|/2$. Prove that there always exists a balanced cut of size at least $|E|/2$.

Hint: Since you want to prove the existence, you might apply randomness.

(ii) Define $d_S(v)$ to be the number of neighbors of vertex v in a set $S \subseteq V$, i.e. $d_S(v) := |\{u \in S : \{v, u\} \in E\}|$. Now, consider the following algorithm. Partition the vertex set V into two arbitrary sets V_1 and V_2. As far as there is a vertex $v \in V_1$ (or $v \in V_2$) for which $d_{V_1}(v) > d_{V_2}(v)$ (resp. $d_{V_2}(v) > d_{V_1}(v)$) move v from V_1 to V_2 (resp. from V_2 to V_1). Prove that this algorithm terminates and generates a cut of size at least $|E|/2$.



$D(n, m)$
$D(\frac{1}{8}n, \frac{3}{4}m)$
$D(\frac{1}{8}n, \frac{3}{8}n)$
$D(\frac{1}{8^2}n, \frac{3}{4}(\frac{3}{4}m))$
$D(\frac{1}{8^2}n, \frac{3}{8^2}n)$
$D(\frac{1}{8^2}n, \frac{3}{4}(\frac{3}{8}n))$
$D(\frac{1}{8^2}n, \frac{3}{8^2}n)$
$T(n^{2/5}, \infty)$

Figure 1: A binary tree of very specific running times that we use in order to prove an upper bound on the actually observed running time of the algorithm.

Exercise 4

In a city there are n houses $h_1, \cdots, h_n$, each of which is in need of a water supply. It costs c_i to build a well at house h_i, and it costs w_{ij} to build a pipe in between houses h_i and h_j. A house can receive water if either there is a well built there or there is some path of pipes to a house with a well. Give an algorithm to find the minimum cost to supply every house with water.

Hint: There is a short solution.