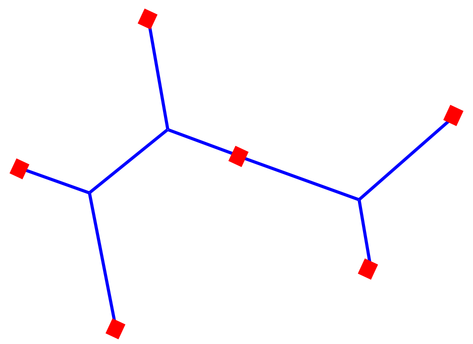
Euclidean Steiner Problem

Instance: n points $x_1, \ldots, x_n$ in the plain.

Problem: Find a shortest network connecting all points x_i.



Important: Use of additional branching points is permitted!

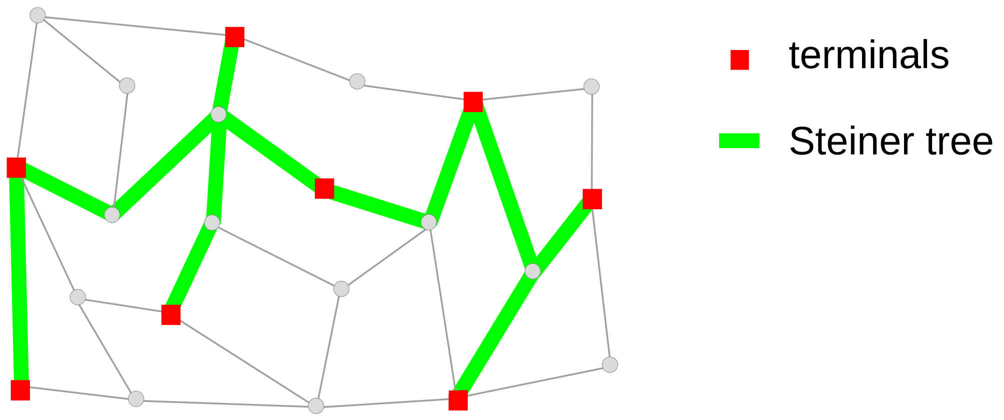
(Otherwise: $\longrightarrow$ MST-problem)

$\longrightarrow$ Named after Jakob Steiner (1796-1863).

Steiner Problem in Graphs

Instance: A graph $G = (V, E)$ and a terminal set $K \subseteq V$.

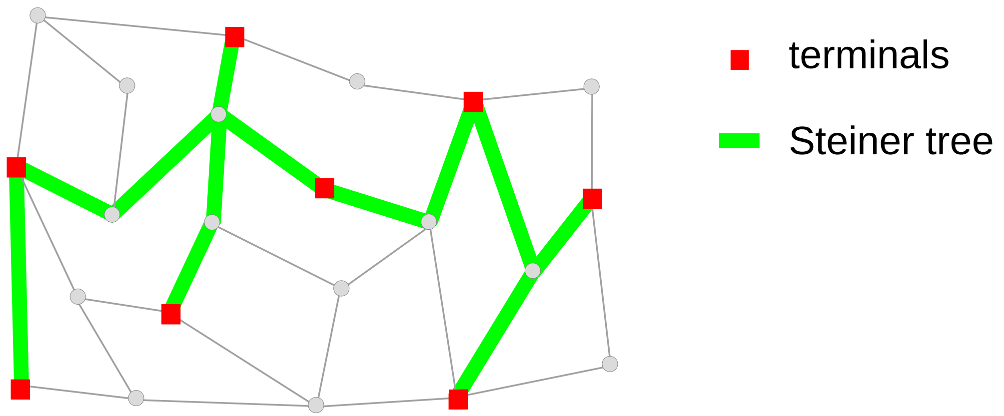
Problem: A minimum Steiner tree for K; I.e., a connected subgraph T such that $K \subseteq V(T)$ and $|E(T)| \stackrel{!}{=}$ minimum.



Steiner Problem in Networks

Instance: A network (weighted graph) $N = (V, E, \ell)$ such that $\ell : E \to \mathbb{R}_{\geq 0}$ and a terminal set $K \subseteq V$.

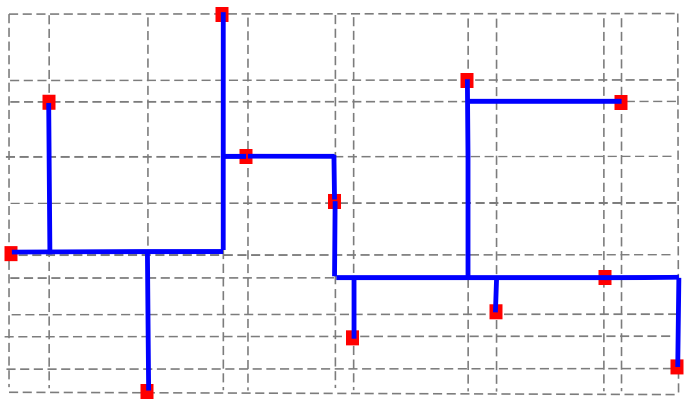
Problem: A minimum Steiner tree for K; I.e., a connected subgraph T such that $K \subseteq V(T)$ and $\ell(T) = \sum_{e \in E(T)} \ell(e) \stackrel{!}{=}$ minimum.



Manhattan Steiner Problem

Instance: n points $x_1, \ldots, x_n$ in the plain.

Problem: Find a shortest (w.r.t. the L_1-norm) network connecting all points x_i.



$\longrightarrow$ Equivalent to the Steiner Problem in (complete) grid graphs [Hanan '66]

Special Cases

$|K| = 2$**:** Shortest Path Problem

$\longrightarrow$ Dijkstra's algorithm: $\mathcal{O}(n \log n + m)$

$K = V$**:** Minimum Spanning Tree Problem

$\longrightarrow$ Prim's algorithm: $\mathcal{O}(n \log n + m)$

Steiner Points

Def: A point v in a Steiner T is called a Steiner point iff $v \notin K$ and $deg(v) \geq 3$.

Exercise: A Steiner tree contains at most $k - 2$ Steiner points.

Exercise: Computation of a minimum Steiner tree is easy if the set of Steiner points is known.

Corollary: A minimum Steiner tree can be found in $\mathcal{O}(n^k)$ time by complete enumeration of all sets of Steiner points.

Computational Complexity

Karp ’72:

The SteinerProblemInNetworks is $\mathcal{NP}$-complete.

Garey, Johnson ’77:

The ManhattanSteinerProblem is $\mathcal{NP}$-complete.

Garey, Graham, Johnson ’77:

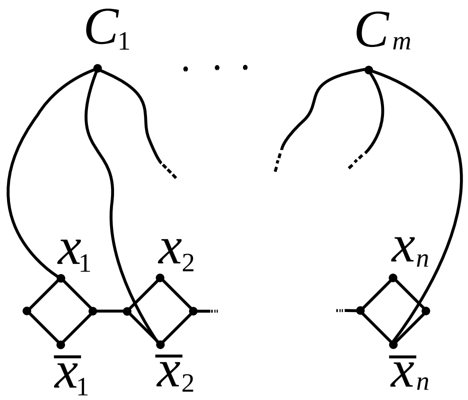
The EuclideanSteinerProblem is $\mathcal{NP}$-hard.

Theorem:

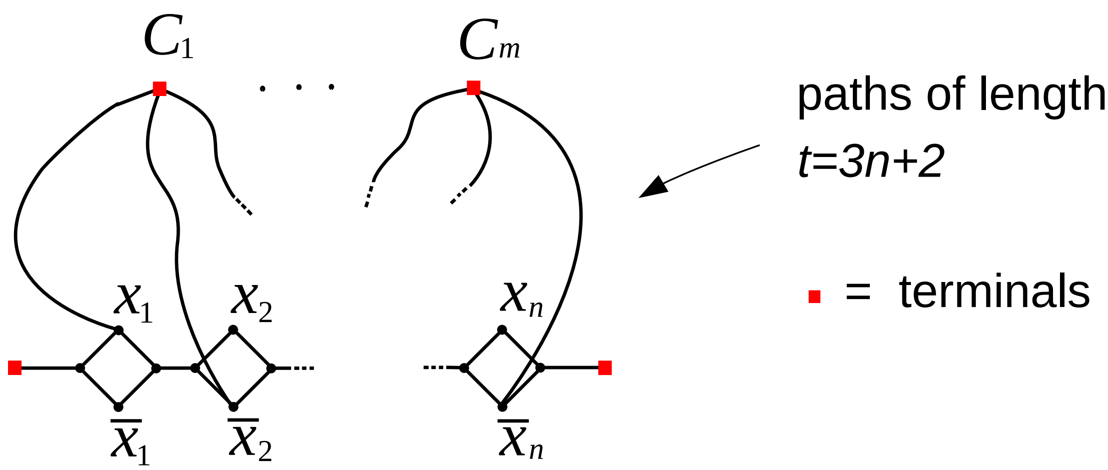
STEINERPROBLEMINPLANARGRAPHS is $\mathcal{NP}$-complete.

Proof [PrSt]: reduction from PLANAR3SAT.

Let I be a 3SAT instance with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$ such that

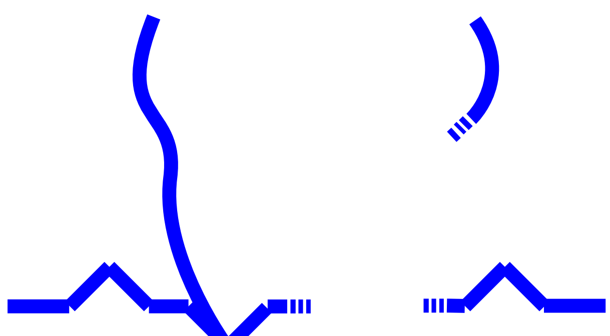


is planar. Consider G_I:

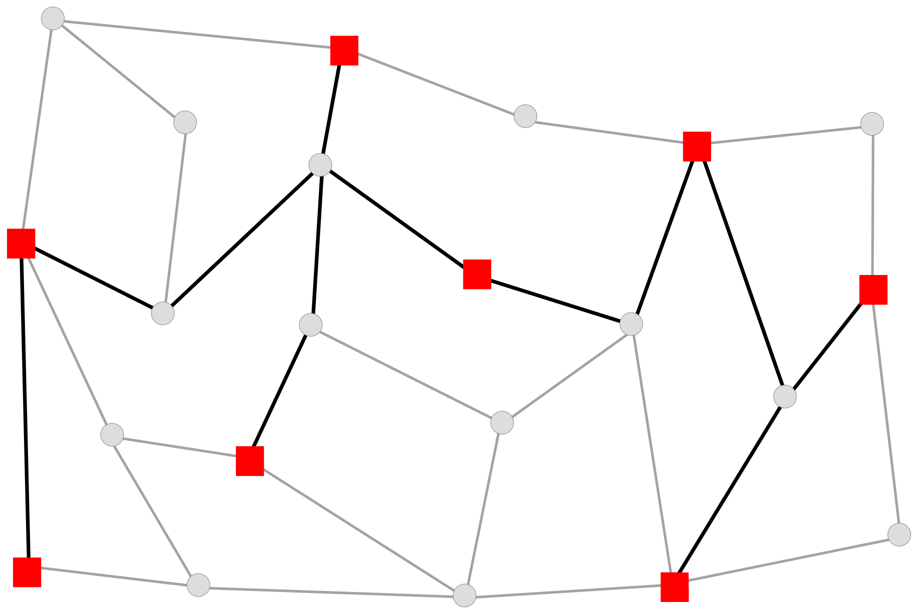


Then: 3SAT instance is satisfiable $\Longleftrightarrow$

G_I contains a Steiner tree of length $\leq 3n + 1 + mt$



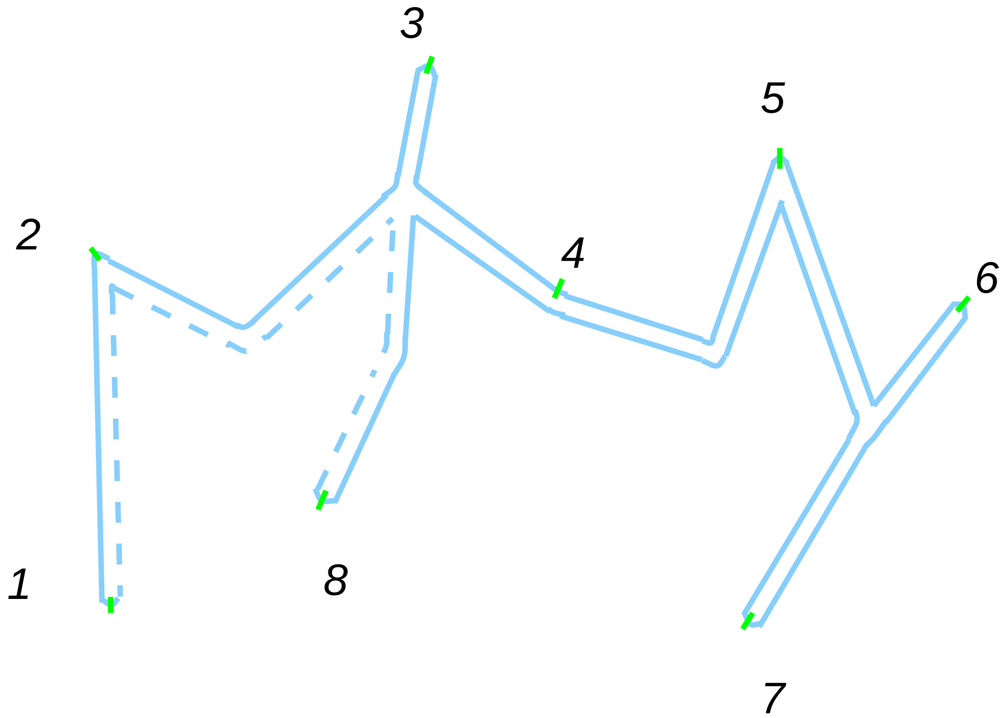
Approximation Ratio 2



- Compute a minimum spanning tree in the distance network induced by K;
- Embed it in the original graph.

[Choukhmane '78, Kou, Markowsky, Berman '81, ...]

Running time: $\mathcal{O}(m + n \log n)$ [Mehlhorn '88]



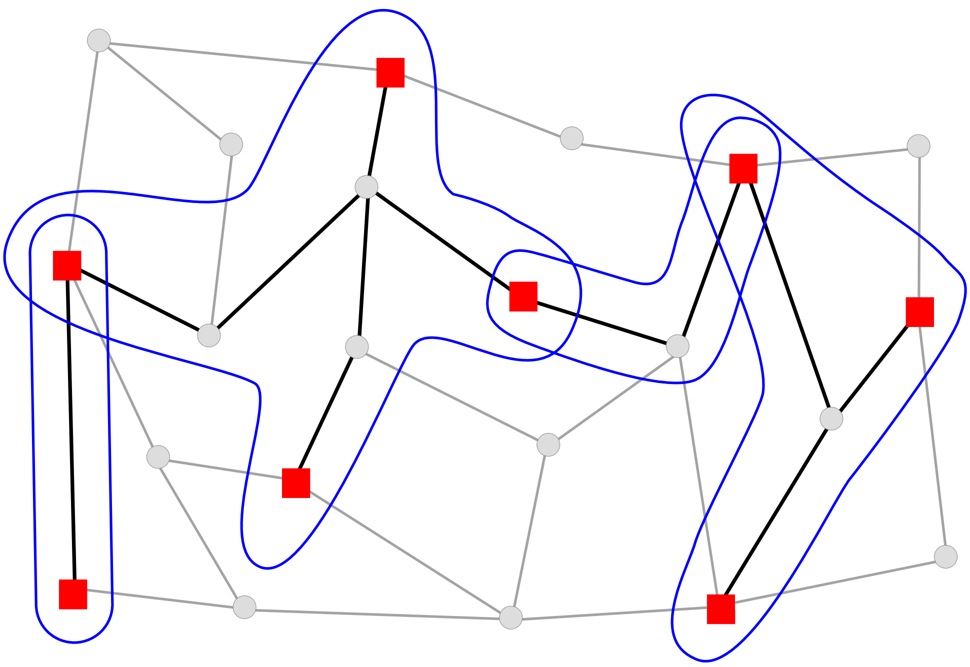
Tour *1-2-3-4-5-6-7-8*

Approximation Ratios – Improvements

Idea: Compute

- not only shortest paths between any two terminals,
- but minimum Steiner trees for all subsets consisting of at most r terminals,

and find a minimum spanning tree in the corresponding hypergraph.



$\Longrightarrow$ optimal solution for $r \geq 4$

Steiner Ratios

For all $r \geq 2$ let

$$H_r[N,K] := (K, F_r; \ell_r) \text{ where}$$
$$F_r := \{f \subseteq K \mid |f| \leq r\},$$
$$\ell_r(f) := smt(N,f) \quad \text{for all } f \in F_r$$

Fact: $H_r[N,K]$ computable in poly. time. (r constant)

Fact: $mst(H_r[N,K]) \geq smt(N,K)$

Let $\rho_r := \inf\{a \mid a \geq \frac{mst(H_r[N,K])}{smt(N,K)} \; \forall \, N, K\}$

Theorem:

$$\rho_2 = 2$$

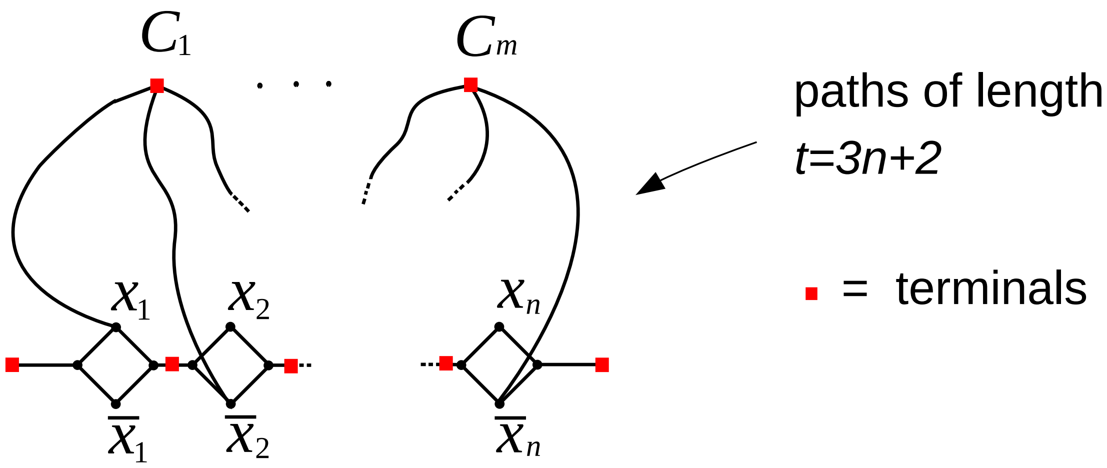
$$\rho_3 = \frac{5}{3}$$ [Zelikovsky '93]

$$\rho_{2^s+r} = \frac{(s+1)2^s + r}{s2^s + r}$$ [Borchers, Du '95]

Corollary: $\rho_r \to 1$ for $r \to \infty$.

Bad News ...

Theorem: MST-problem for $H_r[N,K]$ is $\mathcal{NP}$-complete $\forall r \geq 4$.



[**Fact**: 3SAT remains $\mathcal{NP}$-complete if each literals occurs at most twice.]

Approximation Algorithms

MAIN IDEA: Compute a “good approximation” of a minimum spanning tree.

ZELIKOVSKY’S ALGORITHM $\mathcal{A}_r$

Compute $H_r(N) = (K, F_r, \ell_r)$.

Compute $H_2(N) = (K, E_2, \ell_2)$.

$i := 0$.

while $T_1 \cup \ldots \cup T_i$ is not a connected spanning subgraph of $H_r(N)$ **do**

begin

$i := i + 1$

Choose $T_i \in F_r$ that minimizes

$$f_{i-1}(T_i) := \frac{\ell_r(T_i)}{(mst_2(N;T_1,\ldots,T_{i-1}) - mst_2(N;T_1,\ldots,T_{i-1},T_i))};$$

end;

Compute from $T_1 \cup \ldots \cup T_i$ a Steiner tree T for K in N

State of the Art

Author	Ratio	Running time
Kou, Markowsky, Berman '81, u.a.	2	$\mathcal{O}(n \log n + m)$ [M88]
Zelikovsky '93	$\frac{11}{6} \approx 1.84$	$\mathcal{O}(n^3)$
Berman, Ramaiyer '94	$\frac{16}{9} \approx 1.78$	$\mathcal{O}(n^5)$
	$\rho_2 - \sum_i \frac{\rho_{i-1} - \rho_i}{i-1} \approx 1.734$	$n^k, \quad k \to \infty$
Zelikovsky '95	$1 + \ln 2 \approx 1.69$	$n^k, \quad k \to \infty$
Karpinski, Zelikovsky '95	1.644	$n^k, \quad k \to \infty$
Prömel, St. '97 $^{*)}$	$\frac{5}{3} + \varepsilon \approx 1.667$	$\mathcal{O}(\frac{\log 1/\varepsilon}{\varepsilon} \cdot n^{14} \cdot \log n)$
Hougardy, Prömel '99	≈ 1.59	$n^k, \quad k \to \infty$
Robbins, Zelikovsky '00	$1 + \frac{\ln 3}{2} \approx 1.55$	$n^k, \quad k \to \infty$

MST-Problem in 3-uniform Hypergraphs

(1) Unweighted

Lovász '78:	$\mathcal{O}(n^{17})$
Gabow, Stallmann '86:	$\mathcal{O}(n^4)$
Lovász '79	$\mathcal{O}(m \cdot n^4)$, randomized

(2) Weighted

Camerini, Galbiati, Maffioli '92:	$\tilde{\mathcal{O}}(n^6 \cdot (w_{\text{max}})^2)$, randomized
Prömel, St. '97:	$\mathcal{O}((\log n)^2)$ time,
	$\mathcal{O}(m \cdot n^{7.5} \cdot w_{\text{max}})$ processors, randomized
approximation scheme:	$\mathcal{O}((\log n)^2)$ time,
	$\mathcal{O}(\varepsilon^{-1} \cdot m^2 \cdot n^{8.5})$ processors, randomized

Detour: Pfaffians

Let $A = (a_{ij})$ be $2n \times 2n$ skew-symmetric matrix.

Let $\mathcal{P}$ denote the set of all partitions of $\{1, \ldots, 2n\}$ into two element sets.

For $p = \{\{i_1, i_2\}, \ldots, \{i_{2n-1}, i_{2n}\}\} \in \mathcal{P}$ let

$$\sigma(p) \quad := \quad \text{sign}\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \cdots & i_{2n-1} & i_{2n} \end{pmatrix},$$

$$\rho(p) \quad := \quad \prod_{j=1}^{n} a_{i_{2j-1}\, i_{2j}}.$$

The pfaffian of A is defined as

$$\text{pf}(A) \quad := \sum_{p \in \mathcal{P}} \sigma(p) \cdot \rho(p)$$

Lemma: $\det(A) = [\text{pf}(A)]^2$ and

$\text{pf}(BAB^T) = \det(B) \cdot \text{pf}(A) \qquad \forall B$

Lemma [Lov79, CGM92]:

Let $m \geq n$ and $a_1, b_1, \ldots, a_m, b_m \in \mathbb{R}^{2n}$ and

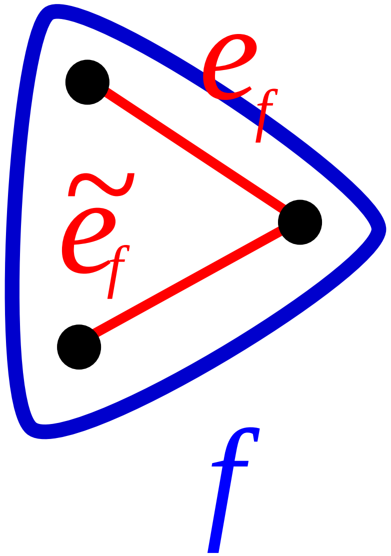
$$A := \sum_{i=1}^{m} x_i (a_i b_i^T - b_i a_i^T).$$

Then: A is a skew-symmetric $2n \times 2n$ matrix and

$$\text{pf}(A) = \sum_{1 \leq i_1 < \ldots < i_n \leq m} x_{i_1} \cdot \ldots \cdot x_{i_n} \cdot \det(\underbrace{a_{i_1} | b_{i_1} \cdots | a_{i_n} | b_{i_n}}_{2n \times 2n \text{ Matrix}}).$$

Back to the MST-Problem ...

Let $H = (V, F)$ be a 3-uniform hypergraph on $2n + 1$ vertices.



For each hyperedge $f \in F$ choose arbitrarily two edges $e_f, \tilde{e}_f \subseteq f$.

Fact: $\{f_1, \ldots, f_n\} \subseteq F$ is a spanning tree in H

$\Longleftrightarrow$ $\{e_{f_1}, \tilde{e}_{f_1}, \ldots, e_{f_n}, \tilde{e}_{f_n}\}$ is a spanning tree in G.

Define $a_f, b_f \in \mathbb{R}^{2n}$ as

$$(a_f)_i := \begin{cases} 1 & \text{if } i \in e_f, \\ 0 & \text{otherwise,} \end{cases} \qquad (b_f)_i := \begin{cases} 1 & \text{if } i \in \tilde{e}_f, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A := \sum_{f \in F} 2^{w(f)} (a_f b_f^T - b_f a_f^T)$$

Observe:

$$\det(a_{i_1} | b_{i_1} \cdots | a_{i_n} | b_{i_n}) = \begin{cases} \pm 1 & \text{if } \{a_{i_1}, b_{i_1}, \cdots, a_{i_n}, b_{i_n}\} \text{ is a spanning tree,} \\ 0 & \text{otherwise,} \end{cases}$$

Lemma: Let H be a 3-uniform hypergraph on $2n+1$ vertices and let $w : F \longrightarrow \mathbb{N}_0$ be a weight function such that H contains a unique spanning tree T_0 of minimum weight, say w_0. Then:

$\det(A) \neq 0$ and 2^{2w_0} is the largest power of 2 that divides $\det(A)$.

In addition, for all $f \in F$ and $A_f = A - 2^{w(f)}(a_f b_f^T - b_f a_f^T)$:

$$f \in T_0 \quad \Longleftrightarrow \quad \frac{\det(A_f)}{2^{2w_0}} \text{ is even.}$$

Idea of the proof:

$$\det(A) = [\mathrm{pf}(A)]^2 = \left[\sum_T 2^{w(T)} \cdot \delta_T\right]^2, \qquad \text{where} \quad \delta_T \in \{-1, +1\}.$$

$\Longrightarrow$ There exists a deterministic algorithm that solves the MST-problem – for hypergraphs with a unique minimum spanning tree.

Idea: $\longrightarrow$ [Mulmuley, Vazirani, Vazirani '87]
Add to the weight of each edge a "small", randomly chosen ε_i. This changes the weight of the spanning trees only "a little", but with probability $\geq 1/2$ the minimum spanning tree is now unique!

Realization: Multiply the weight of each edge with $\Omega(n^4)$ and add a randomly chosen value of size $\mathcal{O}(n^3)$.

$\Longrightarrow$ Algorithm consists essentially of $|F| + 1$ many computations of determinants of $2n \times 2n$ matrices – with entries of (bit) size $\mathcal{O}(n^4 \cdot w_{\text{max}})$.

Approximation Scheme

Idea: $\rightsquigarrow$ Scale ...!

(1) Let $t := \varepsilon \cdot w_{\text{max}}/n$ and $w'(f) = \left\lceil \frac{w(f)}{t} \right\rceil$ for all $f \in F$.

If T is a minimum spanning tree in H', then:

$$w(T) \leq t \cdot w'(T) = t \cdot mst(H') \leq mst(H) + \frac{1}{2}tn = mst(H) + \frac{1}{2}\varepsilon w_{\text{max}}.$$

[OK, if $mst(H) \geq w_{\text{max}}$, but that is not true in general.]

(2) Let $w_1 < \ldots < w_s$ be the different weights in H.

Set $F_i := \{f \in F \mid w(f) \leq w_i\}$ and $H_i = (V, F_i)$.

Then there exists an i_0 such that $w_{i_0} \leq mst(H_{i_0}) = mst(H)$.

$\Longrightarrow$ Apply (1) in parallel to all H_i and return the smallest of all spanning tree.