

**Computational Geometry****Exercise Set 4****HS09**URL: <http://www.ti.inf.ethz.ch/ew/courses/CG09/>**Exercise 1**

Consider the lifting map  $p$  from the plane to the unit paraboloid  $\mathcal{U} = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$  given by  $\ell(x, y) := (x, y, x^2 + y^2)$ . Let  $C$  be a circle in  $\mathbb{R}^2$ . Show that there is a hyperplane  $h_C$  such that

- the lifting  $\ell(C)$  of the circle  $C$  (i.e.  $\{\ell(p) \mid p \in C\}$ ) is the set  $\mathcal{U} \cap h_C$
- the lifting of the interior of the circle  $C$  is the set  $\mathcal{U} \cap h_C^-$  where  $h_C^-$  denotes the lower open halfplane of the hyperplane  $h_C$ .

**Exercise 2**

The Euclidean minimum spanning tree (EMST) of a finite point set  $M \subset \mathbb{R}^2$  is a spanning tree for which the sum of the edge lengths is minimum (among all spanning trees of  $M$ ). Show:

- Every EMST of  $M$  contains a closest pair, i.e. an edge between two points  $p, q \in M$ , that have minimum distance to each other among all point pairs in  $\binom{M}{2}$ .
- Every Delaunay Triangulation of  $M$  contains an EMST of  $M$ .

**\*Exercise 3**

Consider the  $n \times n$  grid. Look at the paths from the lower left corner  $(0, 0)$  to the upper right corner  $(n, n)$  passing on the grid edges and using exactly  $2n$  of them (i.e. the shortest paths).

- How many are there all such paths?
- How many are there such paths, which pass above the diagonal at least once, i.e. use some point with coordinates  $(i, i+1)$  (*Hint*: Find a bijection between such lower-diagonal paths on the  $n \times n$  grid and all paths on the  $(n-1) \times (n+1)$  grid [those you can easily count the same way as in (a)]; for this you can look at the first edge, which goes above the diagonal and after that, switch the direction of the following edges sending the upward edges to the right and right edges upward).
- Conclude that the number of all the paths in  $n \times n$  grid, which stay below the diagonal is  $\frac{1}{n+1} \binom{2n}{n}$ .