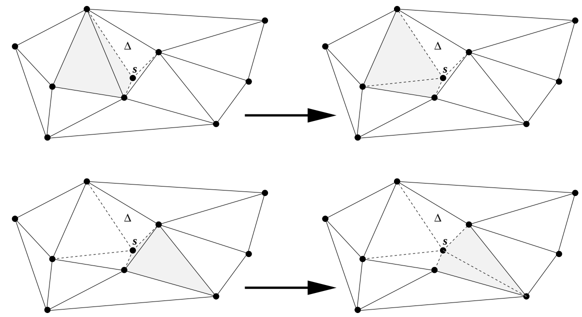
**Randomized** Incremental Construction (RIC) of the Delaunay triangulation



## Problem & Algorithm

Given a set $P$ of $n$ points in $\mathbb{R}^2$ in general position (no 3 on a line, no 4 on a circle), compute the Delaunay triangulation $\mathcal{DT}(P)$ of $P$.



**Algorithm**: *Randomized* Incremental Construction; points are inserted in random order (after the first three far-away points $a$,$b$,$c$)

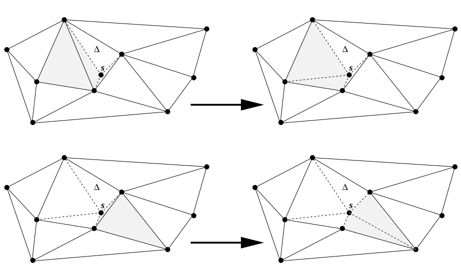
**Def:** $\mathcal{T}_r$ is the Delaunay triangulation after $r$ insertion steps (a random variable).



## $\mathcal{T}_{r-1} \rightarrow \mathcal{T}_r$: Expected Update Cost

- Update cost $= O(1) + O(\text{number of flips})$
- each flip generates a new edge of $\mathcal{T}_r$ incident to the new point $s$
- Update cost =

  $O$(degree of $s$ in $\mathcal{T}_r$)



Example: $s$ has degree 5 in $\mathcal{T}_r$



## Backwards analysis

Run the algorithm backwards (as a movie).

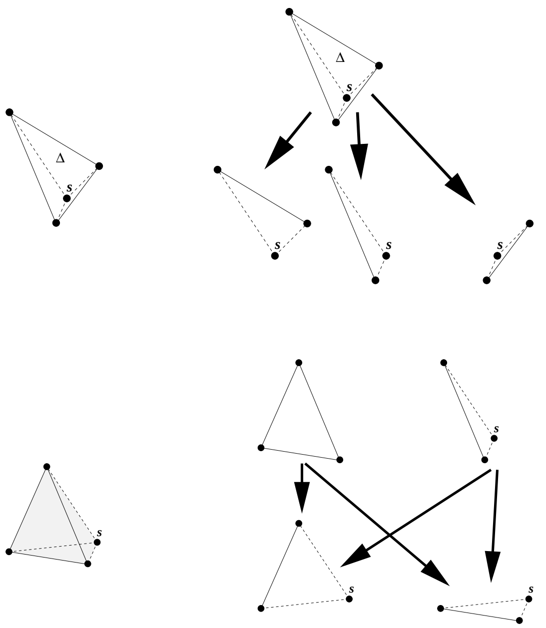
- Random insertion order ≈ random deletion order
- In going from $\mathcal{T}_r$ to $\mathcal{T}_{r-1}$, a *random* point $s \neq a, b, c$ is being deleted
- Its average degree in $\mathcal{T}_r$ is at most

$$\begin{aligned} & \frac{1}{r} \sum_{q \in \mathcal{T}_r \setminus \{a,b,c\}} \deg(q, \mathcal{T}_r) \\ \leq\ & \frac{1}{r} \sum_{q \in \mathcal{T}_r} \deg(q, \mathcal{T}_r) \\ \leq\ & \frac{1}{r}(2(3(r+3)-6)) \approx 6. \end{aligned}$$

⇒ Overall expected update cost is $O(1)$ per step and $O(n)$ in total.

## $\mathcal{T}_{r-1} \to \mathcal{T}_r$: Expected Find Cost

Recall the *History graph* approach






## $\mathcal{T}_{r-1} \to \mathcal{T}_r$: Expected Find Cost

- Every triangle $\Delta$ being traversed while locating $s$ in $\mathcal{T}_{r-1}$
  - is in $\mathcal{T}_k$ for some $k < r$
  - has the point $s$ inside and in particular inside its circumcircle
- Such a pair $(\Delta, s)$ is called a *conflict*.
- Expected Cost of all find steps $= O$(expected total number of conflicts)
- $D(\Delta)$: the three vertices of $\Delta$
- $K(\Delta)$: the points in $\Delta$'s circumcircle
- $x_1, \ldots, x_n$ the insertion order
- $\mathcal{T}(R) = \mathcal{DT}(R \cup \{a, b, c\})$



## Expected number of conflicts

We want to count the total number of conflicts $(s, \Delta)$, i.e.

$$\sum_{r=0}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|.$$

The following are equal: the triangles

- appearing in the step $\mathcal{T}_{r-1} \to \mathcal{T}_r$,
- being in $\mathcal{T}_r$ with $x_r \in D(\Delta)$.

For fixed $R = \{x_1, \ldots, x_r\}$, prob$(x = x_r) = 1/r$ for $x \in R$, so the expected conflict number is

$$\frac{1}{r} \sum_{x \in R} \sum_{\Delta \in \mathcal{T}(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)]$$
$$\leq \frac{3}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|.$$



## An easy but crucial Lemma

**Lemma**.

$$|\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|$$

$$=$$

$$|\mathcal{T}(R)| - |\mathcal{T}(R \cup \{y\})| + \deg(y, \mathcal{T}(R \cup \{y\})).$$

**Proof.** The triangles of $\mathcal{T}(R)$ not in conflict with $y$ are exactly the triangles of $\mathcal{T}(R \cup \{y\})$ that do not have $y$ as a vertex.

## Expected number of conflicts (II)

$K_r$: expected number of new conflicts when $x_r$ is inserted. Since $R$ is random itself, $K_r$ is bounded by

$$\frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|$$

which is

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R)|}_{k_1} -$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R\cup\{y\})|}_{k_2} +$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} \deg(y,\mathcal{T}(R\cup\{y\}))}_{k_3}.$$



## Evaluating $k_1$

$$\begin{aligned} k_1 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R)| \\ &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} |\mathcal{T}(R)| \frac{3}{r} \sum_{y\in X\setminus R} 1 \\ &= \frac{3}{r}(n-r)t_r, \end{aligned}$$

where $t_r$ is the expected number of triangles in $\mathcal{T}_r$.



## Evaluating $k_2$

$$\begin{aligned} k_2 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R\cup\{y\})| \\ &= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X,|R'|=r+1} \frac{3}{r} \sum_{y\in R'} |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X,|R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{3}{r}(r+1)|\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X,|R'|=r+1} \frac{3}{r}(n-r)|\mathcal{T}(R')| \\ &= \frac{3}{r}(n-r)t_{r+1} \\ &= \frac{3}{r+1}(n-(r+1))t_{r+1} + \frac{3n}{r(r+1)}t_{r+1}. \end{aligned}$$



## Evaluating $k_3$

$$\begin{aligned} k_3 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X,|R|=r} \frac{3}{r} \sum_{y\in X\setminus R} \deg(y,\mathcal{T}(R\cup\{y\})) \\ &= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X,|R'|=r+1} \frac{3}{r} \sum_{y\in R'} \deg(y,\mathcal{T}(R')) \\ &\le \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X,|R'|=r+1} \frac{3}{r} 3|\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X,|R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{3}{r} 3|\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X,|R'|=r+1} \frac{n-r}{r+1} \cdot \frac{3}{r} 3|\mathcal{T}(R')| \\ &= \frac{3^2}{r(r+1)}(n-r)t_{r+1} \\ &= \frac{3^2 n}{r(r+1)}t_{r+1} - \frac{3^2}{r+1}t_{r+1}. \end{aligned}$$

## Expected number of conflicts (III)

In step $n$, no conflict is created. In step 0, it's $n$ conflicts. Moreover, $k_1(r+1)$ cancels with the first term of $k_2(r)$, and we get

$$\begin{aligned}\sum_{r=1}^{n-1} K_r &\leq \sum_{r=1}^{n-1}(k_1 - k_2 + k_3) \\ &\leq 3(n-1)t_1 + \\ &\quad 3(3+1)n\sum_{r=1}^{n-1}\frac{t_{r+1}}{r(r+1)} - \\ &\quad 3^2\sum_{r=1}^{n-1}\frac{t_{r+1}}{r+1}.\end{aligned}$$



## Back to Delaunay triangulations

- $t_r \leq 2r - 4 = O(r)$

- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

**Theorem:** The Delaunay triangulation of $n$ points in general position in $\mathbb{R}^2$ can be computed in expected time

$$O(n \log n).$$

**Remark:** The same analyis can be done for other problems (get to at least one more later).