17. Smallest Enclosing Balls: Basics

Lecture on Monday 23rd November, 2009 by Bernd Gärtner <gaertner@inf.ethz.ch>

17.1 Problem Statement and Basics

This problem is related to the linear programming problem, but in a way it is much simpler, since a unique optimal solution always exists.

We let P be a set of n points in $\mathbb{R}^d$. We are interested in finding a closed ball of smallest radius that contains all the points in P, see Figure 17.1.

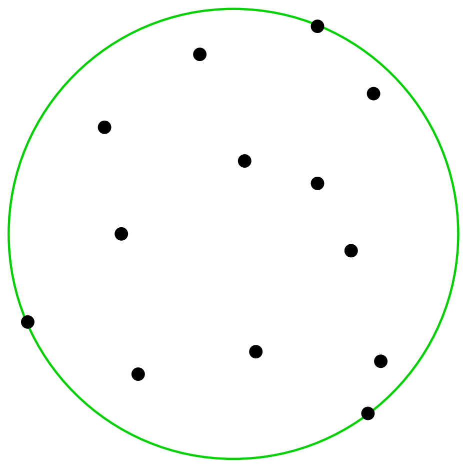

Figure 17.1: *The smallest enclosing ball of a set of points in the plane*

As an "application", imagine a village that wants to build a firehouse. The location of the firehouse should be such that the maximum travel time to any house of the village is as small as possible. If we equate travel time with Euclidean distance, the solution is to place the firehouse in the center of the smallest ball that covers all houses.

17.1.1 Existence

It is not a priori clear that a smallest ball enclosing P exists, but this follows from standard arguments in calculus. As you usually don't find this worked out in papers and textbooks, let us quickly do the argument here.

Fix P and consider the continuous function $\rho : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\rho(c) = \max_{p \in P} \|p - c\|, c \in \mathbb{R}^d$$

Thus, $\rho(c)$ is the radius of the smallest ball centered at c that encloses all points of P. Let q be any point of P, and consider the closed ball

$$B = B(q, \rho(q)) := \{c \in \mathbb{R}^2 \mid \|c - q\| \leq \rho(q)\}.$$

Since B is compact, the function ρ attains its minimum over B at some point c_{opt}, and we claim that c_{opt} is the center of a smallest enclosing ball of P. For this, consider any center $c \in \mathbb{R}^2$. If $c \in B$, we have $\rho(c) \geq \rho(c_{opt})$ by optimality of c_{opt} in B, and if $c \notin B$, we get $\rho(c) \geq \|c - q\| > \rho(q) \geq \rho(c_{opt})$ since $q \in B$. In any case, we get $\rho(c) \geq \rho(c_{opt})$, so c_{opt} is indeed a best possible center.

17.1.2 Uniqueness

Can it be that there are two distinct smallest enclosing balls of P? No, and to rule this out, we use the concept of *convex combinations* of balls. Let $B = B(c, \rho)$ be a closed ball with center c and radius $\rho > 0$. We define the *characteristic function* of B as the function $f_B : \mathbb{R}^2 \to R$ given by

$$f_B(x) = \frac{\|x - c\|^2}{\rho^2}, \quad x \in \mathbb{R}^2.$$

The name characteristic function comes from the following easy

Observation 17.1 *For* $x \in \mathbb{R}^2$*, we have*

$$x \in B \quad \Leftrightarrow \quad f_B(x) \leq 1.$$

Now we are prepared for the convex combination of balls.

Lemma 17.2 *Let* $B_0 = B(c_0, \rho_0)$ *and* $B_1 = (c_1, \rho_1)$ *be two distinct balls with characteristic functions* f_{B_0} *and* f_{B_1}*. For* $\lambda \in (0, 1)$*, consider the function* f_λ *defined by*

$$f_\lambda(x) = (1 - \lambda) f_{B_0}(x) + \lambda f_{B_1}(x).$$

Then the following three properties hold.

(i) f_λ *is the characteristic function of a ball* $B_\lambda = (c_\lambda, \rho_\lambda)$*.* B_λ *is called a (proper)* convex combination *of* B_0 *and* B_1*, and we simply write*

$$B_\lambda = (1 - \lambda) B_0 + \lambda B_1.$$

(ii) $B_\lambda \supseteq B_0 \cap B_1$ *and* $\partial B_\lambda \supseteq \partial B_0 \cap \partial B_1$.

(iii) $\rho_\lambda < \max(\rho_0, \rho_1)$.

A proof of this lemma requires only elementary calculations and can be found for example in the PhD thesis of Kaspar Fischer [1]. Here we will just explain what the lemma means. The family of balls $B_\lambda, \lambda \in (0, 1)$ "interpolates" between the balls B_0 and B_1: while we increase λ from 0 to 1, we continuously transform B_0 into B_1. All intermediate balls B_λ "go through" the intersection of the original ball boundaries (a sphere of dimension $d - 2$). In addition, each intermediate ball contains the intersection of the original balls. This is property (ii). Property (iii) means that all intermediate balls are smaller than the larger of B_0 and B_1. Figure 17.2 illustrates the situation.

Using this lemma, we can easily prove the following

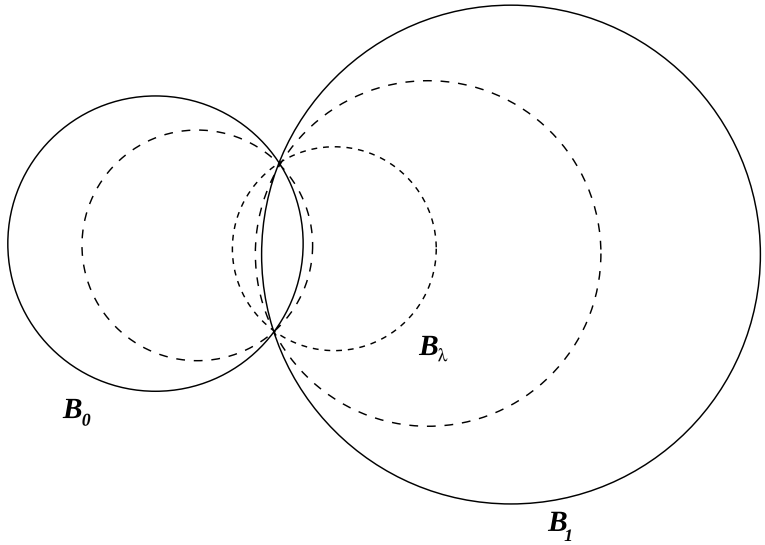

Figure 17.2: *Convex combinations* B_λ *of two balls* B_0, B_1

Theorem 17.3 *Given a finite point set* $P \subseteq \mathbb{R}^d$*, there exists a* unique *ball of smallest radius that contains* P*. We will denote this ball by* $B(P)$*.*

Proof. If $P = \{p\}$, the unique smallest enclosing ball is $\{p\}$. Otherwise, any smallest enclosing ball of P has positive radius ρ_{opt}. Assume there are two distinct smallest enclosing balls B_0, B_1. By Lemma 17.2, the ball

$$B_{\frac{1}{2}} = \frac{1}{2}B_0 + \frac{1}{2}B_1$$

is also an enclosing ball of P (by property (ii)), but it has smaller radius than ρ_{opt} (by property (iii), a contradiction to B_0, B_1 being smallest enclosing balls. □

17.1.3 Bases

When you look at the example of Figure 17.1, you notice that only three points are essential for the solution, namely the ones on the boundary of the smallest enclosing ball. Removing all other points from P would not change the smallest enclosing ball. Even in cases where more points are on the boundary, it is always possible to find a subset of at most three points (in the $\mathbb{R}^2$ case) with the same smallest enclosing ball. This is again a consequence of *Helly's Theorem* (Theorem 15.5).

Theorem 17.4 *Let* $P \subseteq \mathbb{R}^d$ *be a finite point set. There is a subset* $S \subseteq P, |S| \leq d+1$ *such that* $B(P) = B(S)$*.*

Proof. If $|P| < d+1$, we may choose $S = P$. Otherwise, let ρ_{opt} be the radius of the smallest enclosing ball $B(P)$ of $P = \{p_1, \ldots, p_n\}$. Now define

$$C_i = \{x \in \mathbb{R}^d : \|x - p_i\| < \rho_{opt}\}, \quad i = 1, \ldots, n$$

to be the open ball around p_i with radius ρ_{opt}. We know that the common intersection of all the C_i is empty, since any point in the intersection would be a center of an enclosing ball of P with radius smaller than ρ_{opt}. Moreover, the C_i are convex, so Helly's Theorem implies that there is a subset S of $d+1$ points whose C_i's also have an empty common intersection. For this set S, we therefore have no enclosing ball of radius smaller than ρ_{opt} either. Hence, $B(S)$ has radius at leat ρ_{opt}; but since $S \subseteq P$, the radius of $B(S)$ must also be at most ρ_{opt}, and hence it is equal to ρ_{opt}. But then $B(S) = B(P)$ follows, since otherwise, both $B(P)$ and $B(S)$ would be smallest enclosing balls of S, a contradiction. □

The previous theorem motivates the following

Definition 17.5 *Let* $P \subseteq \mathbb{R}^d$ *be a finite point set. A* basis *of* P *is an inclusion-minimal subset* $S \subseteq P$ *such that* $B(P) = B(S)$.

It follows that any basis of P has size at most $d+1$. If the points are in general position (no $k+3$ on a common k-dimensional sphere), then P has a unique basis, and this basis is formed by the set of points on the boundary of $B(P)$.

17.1.4 The trivial algorithm

Theorem 17.4 immediately implies the following (rather inefficient) algorithm for computing $B(P)$: for every subset $S \subseteq P, |S| \leq d+1$, compute $B(S)$ (in fixed dimension d, this can be done in constant time), and return the one with largest radius.

Indeed, this works: for all $S \subseteq P$, the radius of $B(S)$ is at most that of $B(P)$, and there must be at least one $S, |S| \leq d+1$ (a basis of P) with $B(S) = B(P)$. It follows that the ball $B(T)$ being returned has the same radius as $B(P)$ and is therefore equal by $T \subseteq P$.

Assuming that d is fixed, the runtime of this algorithm is

$$O\left(\sum_{i=0}^{d+1} \binom{n}{i}\right) = O(n^{d+1}).$$

If $d = 2$ (the planar case), the trivial algorithm has runtime $O(n^3)$. In the next section, we discuss an algorithm that is substantially better than the trivial one in any dimension.

References

[1] Kaspar Fischer, *Smallest enclosing balls of balls. Combinatorial Structure and Algorithms*, Ph.D. thesis, ETH Zürich, 2005.