# 18. Smallest Enclosing Balls: The Swiss Algorithm

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## 18.1 The Swiss Algorithm

The name of this algorithm comes from the democratic way in which it works. Let us describe it for the problem of locating the firehouse in a village.

Here is how it is done the Swiss way: a meeting of all $n$ house owners is scheduled, and every house owner is asked to put a slip of paper with his/her name on it into a voting box. Then a constant number $r$ (to be determined later) of slips is drawn at random from the voting box, and the selected house owners have the right to negotiate a location for the firehouse among them. They naturally do this in a selfish way, meaning that they agree on the center of the smallest enclosing ball $D$ of just *their* houses as the proposed location.

The house owners that were not in the selected group now fall into two classes: those that are happy with the proposal, and those that are not. Let's say that a house owner $p$ is happy if and only if his/her house is also covered by $D$. In other words, $p$ is happy if and only if the proposal would have been the same with $p$ as an additional member of the selected group.

Now, the essence of Swiss democracy is to negotiate until everybody is happy, so as long as there are any unhappy house owners at all, the whole process is repeated. But in order to give the unhappy house owners a higher chance of influencing the outcome of the next round, their slips in the voting box are being doubled before drawing $r$ slips again. Thus, there are now two slips for each unhappy house owner, and one for each happy one.

After round $k$, a house owner that has been unhappy after exactly $i$ of the $k$ rounds has therefore $2^i$ slips in the voting box for the next round.

The obvious question is: how many rounds does it take until all house owners are happy? So far, it is not even clear that the meeting ever ends. But Swiss democracy is efficient, and we will see that the meeting actually ends after an expected number of $O(\log n)$ rounds. We will do the analysis for general dimension $d$ (just imagine the village and its houses to lie in $\mathbb{R}^d$).

### 18.1.1 The Forever Swiss Algorithm

In the analysis, we want to argue about a fixed round $k$, but the above algorithm may never get to this round (for large $k$, we even strongly hope that it never gets there). But for the purpose of the analysis, we formally let the algorithm continue even if everybody is happy after some round (in such a round, no slips are being doubled).

We call this extension the Forever Swiss Algorithm. A round is called *controversial* if it renders at least one house owner unhappy.

**Definition 18.1**

*(i) Let $m_k$ be the random variable for the total number of slips after round $k$ of the Forever Swiss Algorithm. We set $m_0 = n$, the initial number of slips.*

*(ii) Let $C_k$ be the event that the first $k$ rounds in the Forever Swiss Algorithm are controversial.*

### 18.1.2 A lower bound for $E(m_k)$

Let $S \subseteq P$ be a basis of $P$. Recall that this means that $S$ is inclusion-minimal with $B(S) = B(P)$.

**Observation 18.2** *After every controversial round, there is an unhappy house owner in $S$.*

**Proof.** Let $Q$ be the set of selected house owners in the round. Let us write $B \geq B'$ for two balls if the radius of $B$ is at least the radius of $B'$.

If all house owners in $S$ were happy with the outcome of the round, we would have

$$B(Q) = B(Q \cup S) \geq B(S) = B(P) \geq B(Q),$$

where the inequalities follow from the corresponding superset relations. The whole chain of inequalities would then imply that $B(P)$ and $B(Q)$ have the same radius, meaning that they must be equal (we had this argument before). But then, *nobody* would be unhappy with the round, a contradiction to the current round being controversial. □

Since $|S| \leq d + 1$ by Theorem 17.4, we know that after $k$ rounds, some element of $S$ must have doubled its slips at least $k/(d + 1)$ times, given that all these rounds were controversial. This implies the following lower bound on the total number $m_k$ of slips.

**Lemma 18.3**

$$E(m_k) \geq 2^{k/(d+1)} \operatorname{prob}(C_k), \quad k \geq 0.$$

**Proof.** By the partition theorem of conditional expectation, we have

$$E(m_k) = E(m_k \mid C_k) \operatorname{prob}(C_k) + E(m_k \mid \overline{C_k}) \operatorname{prob}(\overline{C_k}) \geq 2^{k/(d+1)} \operatorname{prob}(C_k).$$

□

### 18.1.3 An upper bound for $E(m_k)$

The main step is to show that the expected increase in the number of slips from one round to the next is bounded.

**Lemma 18.4** *For all* $m \in \mathbb{N}$ *and* $k > 0$,

$$E(m_k \mid m_{k-1} = m) \leq m\left(1 + \frac{d+1}{r}\right).$$

**Proof.** Since exactly the "unhappy slips" are being doubled, the expected increase in the total number of slips equals the expected number of unhappy slips, and this number is

$$\frac{1}{\binom{m}{r}} \sum_{|R|=r} \sum_{h \notin R} [h \text{ is unhappy with } R] = \frac{1}{\binom{m}{r}} \sum_{|Q|=r+1} \sum_{h \in Q} [h \text{ is unhappy with } Q \setminus \{h\}]. \quad (18.5)$$

**Claim:** Every $(r+1)$-element subset $Q$ contains at most $d+1$ slips such that $h$ is unhappy with $Q \setminus \{h\}$.

To see the claim, choose a basis $S$, $|S| \leq d+1$, of the ball resulting from drawing slips in $Q$. Only the removal of a slip $h$ belonging to some house owner $p \in S$ can have the effect that $Q$ and $Q \setminus \{h\}$ lead to different balls. Moreover, in order for this to happen, the slip $h$ must be the *only* slip of the owner $p$. This means that at most one slip $h$ per owner $p \in S$ can cause $h$ to be unhappy with $Q \setminus \{h\}$.

We thus get

$$\frac{1}{\binom{m}{r}} \sum_{|R|=r} \sum_{h \notin R} [h \text{ is unhappy with } R] \leq (d+1)\frac{\binom{m}{r+1}}{\binom{m}{r}} = (d+1)\frac{m-r}{r+1} \leq (d+1)\frac{m}{r}. \quad (18.6)$$

By adding $m$, we get the new expected total number $E(m_k \mid m_{k-1} = m)$ of slips. □

From this, we easily get our actual upper bound on $E(m_k)$.

**Lemma 18.7**

$$E(m_k) \leq n\left(1 + \frac{d+1}{r}\right)^k, \quad k \geq 0.$$

**Proof.** We use induction, where the case $k = 0$ follows from $m_0 = n$. For $k > 0$, the partition theorem of conditional expectation gives us

$$\begin{aligned} E(m_k) &= \sum_{m \geq 0} E(m_k \mid m_{k-1} = m)\ \text{prob}(m_{k-1} = m) \\ &\leq \left(1 + \frac{d+1}{r}\right) \sum_{m \geq 0} m\ \text{prob}(m_{k-1} = m) \\ &= \left(1 + \frac{d+1}{r}\right) E(m_{k-1}). \end{aligned}$$

Applying the induction hypothesis to $E(m_{k-1})$, the lemma follows. □

### 18.1.4 Putting it together

Combining Lemmas 18.3 and 18.7, we know that

$$2^{k/(d+1)}\,\mathrm{prob}(C_k) \le n\left(1+\frac{d+1}{r}\right)^k,$$

where $C_k$ is the event that there are $k$ or more controversial rounds.

This inequality gives us a useful upper bound on $\mathrm{prob}(C_k)$, because the left-hand side power grows faster than the right-hand side power as a function of $k$, given that $r$ is chosen large enough.

Let us choose $r = c(d+1)^2$ for some constant $c > \log_2 e \approx 1.44$. We obtain

$$\mathrm{prob}(C_k) \le n\left(1+\frac{1}{c(d+1)}\right)^k / 2^{k/(d+1)} \le n2^{k\log_2 e/(c(d+1))-k/(d+1)},$$

using $1 + x \le e^x = 2^{x\log_2 e}$ for all $x$. This further gives us

$$\mathrm{prob}(C_k) \le n\alpha^k, \tag{18.8}$$

$$\alpha = \alpha(d,c) = 2^{(\log_2 e - c)/c(d+1)} < 1.$$

This implies the following tail estimate.

**Lemma 18.9** *For any $\beta > 1$, the probability that the Forever Swiss Algorithm performs at least $\lceil \beta \log_{1/\alpha} n\rceil$ controversial rounds is at most*

$$1/n^{\beta-1}.$$

**Proof.** The probability for at least this many controversial rounds is at most

$$\mathrm{prob}(C_{\lceil \beta\log_{1/\alpha} n\rceil}) \le n\alpha^{\lceil \beta\log_{1/\alpha} n\rceil} \le n\alpha^{\beta\log_{1/\alpha} n} = nn^{-\beta} = 1/n^{\beta-1}.$$

□

In a similar way, we can also bound the expected number of controversial rounds of the Forever Swiss Algorithm. This also bounds the expected number of rounds of the Swiss Algorithm, because the latter terminates upon the first non-controversial round.

**Theorem 18.10** *For any fixed dimension $d$, and with $r = \lceil \log_2 e(d+1)^2\rceil > \log_2 e(d+1)^2$, the Swiss algorithm terminates after an expected number of $O(\log n)$ rounds.*

**Proof.** By definition of $C_k$ (and using $E(X) = \sum_{m\ge 1} \mathrm{prob}(X \ge m)$ for a random variable with values in $\mathbb{N}$), the expected number of rounds of the Swiss Algorithm is

$$\sum_{k\ge 1} \mathrm{prob}(C_k).$$

For any $\beta > 1$, we can use (18.8) to bound this by

$$\begin{aligned}\sum_{k=1}^{\lceil \beta \log_{1/\alpha} n\rceil - 1} 1 + n \sum_{k=\lceil \beta \log_{1/\alpha} n\rceil}^{\infty} \alpha^k &= \lceil \beta \log_{1/\alpha} n\rceil - 1 + n\frac{\alpha^{\lceil \beta \log_{1/\alpha} n\rceil}}{1-\alpha} \\ &\leq \beta \log_{1/\alpha} n + n\frac{\alpha^{\beta \log_{1/\alpha} n}}{1-\alpha} \\ &= \beta \log_{1/\alpha} n + \frac{n^{-\beta+1}}{1-\alpha} \\ &= \beta \log_{1/\alpha} n + o(1).\end{aligned}$$

□

What does this mean for $d = 2$? In order to find the location of the firehouse efficiently (meaning in $O(\log n)$ rounds), 13 slips should be drawn in each round. The resulting constant of proportionality in the $O(\log n)$ bound will be pretty high, though. To reduce the number of rounds, it may be advisable to choose $r$ somewhat larger.

Since a single round can be performed in time $O(n)$ for fixed $d$, we can summarize our findings as follows.

**Theorem 18.11** *Using the Swiss Algorithm, the smallest enclosing ball of a set of $n$ points in fixed dimension $d$ can be computed in expected time $O(n \log n)$.*

The Swiss algorithm is a simplification of an algorithm by Clarkson [1, 2].

The bound of the previous Theorem already compares farovably with the bound of $O(n^{d+1})$ for the trivial algorithm, see Section 17.1.4, but it does not stop here. We can even solve the problem in expected linear time $O(n)$, by using an adaptation of Seidel's linear programming algorithm [3].

## 18.2 Smallest Enclosing Balls in the Manhattan Distance

We can also compute smallest enclosing balls w.r.t. distances other than the Euclidean distance. In general, if $\delta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a metric, the smallest enclosing ball problem with respect to $\delta$ is the following.

Given $P \subseteq \mathbb{R}^d$, find $c \in \mathbb{R}^d$ and $\rho \in \mathbb{R}$ such that

$$d(c, p) \leq \rho, \quad p \in P,$$

and $\rho$ is as small as possible.

For example, if $d(x, y) = \|x - y\|_\infty = \max_{i=1}^d |x_i - y_i|$, the problem is to find a smallest axis-parallel cube that contains all the points. This can be done in time $O(d^2 n)$ by finding the smallest enclosing box. The largest side-length of the box corresponds to the largest extent of the point set in any of the coordinate directions; to obtain a smallest enclosing cube, we simply extend the box along the other directions until all side lengths are equal.

A more interesting case is $d(x,y) = ||x-y||_1 = \sum_{i=1}^{d} |x_i - y_i|$. This is the *Manhattan distance*. There, the problem can be written as

$$\begin{array}{ll} \text{minimize} & \rho \\ \text{subject to} & \sum_{i=1}^{d} |p_{ji} - c_i| \leq \rho, \quad j = 1, \ldots, n. \end{array}$$

where $p_j$ is the j-th point and $p_{ji}$ it's i-th coordinate Geometrically, the problem is now that of finding a smallest *cross polytope* (generalized octahedron) that contains the points. Algebraically, we can reduce it to a linear program, as follows.

We replace all $|p_{ji} - c_i|$ by new variables $y_{ji}$ and add the additional constraints $y_{ji} \geq p_{ji} - c_i$ and $y_{ji} \geq c_i - pji$. The problem now is a linear program.

$$\begin{array}{llll} \text{minimize} & \rho & & \\ \text{subject to} & \sum_{i=1}^{d} y_{ji} & \leq & \rho, \quad j = 1, \ldots, n \\ & y_{ji} & \geq & p_{ji} - c_i, \quad \forall i, j \\ & y_{ji} & \geq & c_i - p_{ji}, \quad \forall i, j. \end{array}$$

The claim is that the solution to this linear program also solves the original problem. For this, we need to observe two things: first of all, every optimal solution $(\tilde{c}, \tilde{\rho})$ to the original problem induces a feasible solution to the LP with the same value (simply set $y_{ji} := |p_{ji} - \tilde{c}_i|$), so the LP solution has value equal to $\tilde{\rho}$ or better. The second is that every optimal solution $((\tilde{y}_{ji})_{i,j}, \tilde{\rho})$ to the LP induces a feasible solution to the orginal problem with the same value: by $\sum_{i=1}^{d} \tilde{y}_{ji} \leq \tilde{\rho}$ and $\tilde{y}_{ji} \geq |p_{ji} - c_i|$, we also have $\sum_{i=1}^{d} |p_{ji} - c_i| \leq \tilde{\rho}$. This means, the original problem has value $\tilde{\rho}$ or better. From these two observations it follows that both problems have the same optimal value $\rho_{opt}$, and an LP solution of this value yields a smallest enclosing ball of P w.r.t. the Manhatten distance.

## References

[1] K. L. Clarkson, A Las Vegas algorithm for linear programming when the dimension is small, in: *Proc. 29th Annu. IEEE Sympos. Found. Comput. Sci.*, 1988, 452–456.

[2] K. L. Clarkson, Las Vegas algorithms for linear and integer programming, *J. ACM* **42** (1995), 488–499.

[3] Emo Welzl, Smallest enclosing disks (balls and ellipsoids), in: *New Results and New Trends in Computer Science* (H. Maurer, ed.), volume 555 of *Lecture Notes Comput. Sci.*, Springer-Verlag, 1991, 359–370.