7. Voronoi Diagrams

Lecture on Monday 12^{th} October, 2009 by Michael Hoffmann <hoffmann@inf.ethz.ch>

7.1 Post Office Problem

Suppose there are n post offices $p_1, \ldots p_n$ in a city. Someone who is located at a position q within the city would like to know which post office is closest to him. Modeling the city as a planar region, we think of $p_1, \ldots p_n$ and q as points in the plane. Denote the set of post offices by $P = \{p_1, \ldots p_n\}$. While the locations of post offices are known and do not change so frequently, we do not know in advance for which—possibly many—query locations the closest post office is to be found. Therefore, our long term goal is to come up with a data structure on top of P that allows to answer any possible query efficiently. The basic idea is to apply a so-called *locus approach*: we partition the query space into regions on which is the answer is the same. In our case, this amounts to partition the plane into regions such that for all points within a region the same point from P is closest (among all points from P).

As a warmup, consider the problem for two post offices $p_i, p_j \in P$. For which query locations is the answer p_i rather than p_j? This region is bounded by the bisector of p_i and p_j, that is, the set of points which have the same distance to both points.

Proposition 7.1 *For any two distinct points in $\mathbb{R}^d$ the bisector is a hyperplane, that is, in $\mathbb{R}^2$ it is a line.*

Proof. Let $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ be two points in $\mathbb{R}^d$. The bisector of p and q consists of those points $x = (x_1, \ldots, x_d)$ for which

$$||p - x|| = ||q - x|| \iff ||p - x||^2 = ||q - x||^2 \iff ||p||^2 - ||q||^2 = 2(p - q)x .$$

As p and q are distinct, this is the equation of a hyperplane. □

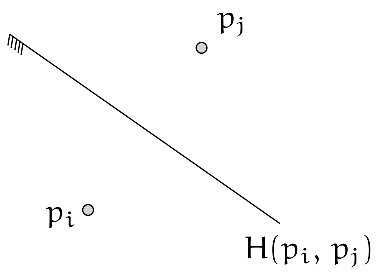


Figure 7.1: *The bisector of two points.*

Denote by $H(p_i, p_j)$ the closed halfplane bounded by the bisector of p_i and p_j that contains p_i.

7.2 Voronoi Diagram

In the following we work with a set $P = \{p_1, \ldots, p_n\}$ of points in $\mathbb{R}^2$.

Definition 7.2 (Voronoi cell) *For* $p_i \in P$ *denote the* **Voronoi cell** $V_P(i)$ *of* p_i *by*

$$V_P(i) := \left\{ q \in \mathbb{R}^2 \,\middle|\, \|q - p_i\| \le \|q - p\| \text{ for all } p \in P \right\}$$

Proposition 7.3

$$V_P(i) = \bigcap_{j \neq i} H(p_i, p_j)\ .$$

Proof. For $j \neq i$ we have $\|x - p_i\| \le \|x - p_j\| \iff x \in H(p_i, p_j)$. □

Corollary 7.4 $V_P(i)$ *is non-empty and convex.*

Proof. According to Proposition 7.3 $V_P(i)$ is the intersection of a finite number of halfspaces and hence convex. $V_P(i) \neq \emptyset$ because $p_i \in V_P(i)$. □

Observe that every point of the plane lies in some Voronoi cell but no point lies in the interior of two Voronoi cells. Therefore these cells form a subdivision of the plane.

Definition 7.5 (Voronoi Diagram) *The Voronoi Diagram* $VD(P)$ *of a set* $P = \{p_1, \ldots, p_n\}$ *of points in* $\mathbb{R}^2$ *is the subdivision of the plane induced by the Voronoi cells* $V_P(i)$*, for* $i = 1, \ldots, n$.
Denote by $VV(P)$ *the set of vertices, by* $VE(P)$ *the set of edges, and by* $VR(P)$ *the set of regions (faces) of* $VD(P)$.

Lemma 7.6 *For every vertex* $v \in VV(P)$ *the following statements hold.*

a) v *is the common intersection of at least three edges from* $VE(P)$*;*

b) v *is incident to at least three regions from* $VR(P)$*;*

c) v *is the center of a circle* $C(v)$ *through at least three points from* P *such that*

d) $\mathrm{Int}(C(v)) \cap P = \emptyset$.

Proof. Consider a vertex $v \in VV(P)$. As all Voronoi cells are convex, $k \ge 3$ of them must be incident to v. This proves Part a) and b). Without loss of generality let these cells be $V_P(i)$, for $1 \le i \le k$. Denote by e_i, $1 \le i \le k$, the edge incident to v that bounds $V_P(i)$ and $V_P((i \bmod k) + 1)$. For any $i = 1, \ldots, k$ we have $v \in e_i \Rightarrow |v - p_i| = |v - p_{(i \bmod k)+1}|$. In other words, $p_1, p_2, \ldots, p_k$ are cocircular, which proves Part c).

Part d): Suppose there is a point $p_\ell \in \mathrm{Int}(C(v))$. Then v is closer to p_ℓ than to any of $p_1, \ldots, p_k$, in contradiction to the fact that v is incident to all of $V_P(1), \ldots, V_P(k)$. □

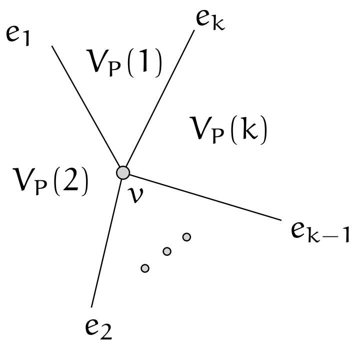


Figure 7.2: *Voronoi regions around* v*.*

Corollary 7.7 *If no four points in* P *are cocircular, for every vertex* $v \in \mathrm{VV}(P)$ *the following statements hold.*

a) v *is the common intersection of exactly three edges from* $\mathrm{VE}(P)$*;*

b) v *is incident to exactly three regions from* $\mathrm{VR}(P)$*;*

c) v *is the center of a circle* $C(v)$ *through exactly three points from* P *such that*

d) $\mathrm{Int}(C(v)) \cap P = \emptyset$. □

Lemma 7.8 *Suppose that no three points in* P *are collinear.*[1] *There is an unbounded Voronoi edge bounding* $V_P(i)$ *and* $V_P(j)$ $\iff$ $\overline{p_i p_j}$ *is an edge of* $\mathrm{conv}(P)$*.*

Proof. Consider the family of all circles through p_i and p_j and let

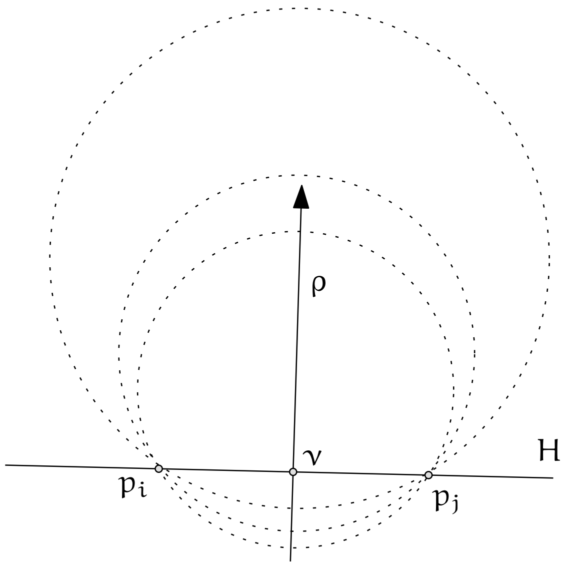
$$\mathcal{C} = \{c_k \mid c_k \text{ is center of the circle through } p_i, p_j, \text{ and } p_k\}.$$

Let H be some closed halfplane through p_i and p_j. As all points from $\mathcal{C}$ lie on the bisector $b_{i,j}$ of p_i and p_j, there is a natural linear order, starting from the point that is furthest away from H up to the point that is furthest inside H. (Clearly there may be no point from $\mathcal{C}$ inside H or no point outside of H, but the order is always well defined.) Let c_k be the maximum from $\mathcal{C}$ according to this order, that is, the point from $\mathcal{C}$ that is located furthest inside H.

The ray ρ starting from c_k along $b_{i,j} \cap H$ is a Voronoi edge. $\iff$ For any $c \in \rho$ there is no point from P closer to c than p_i and p_j. $\iff$ For any $c \in \rho$ the circle centered at c through p_i and p_j does not contain any point from P in its interior. $\iff$ $H \cap P = \{p_i, p_j\}$.

The last statement implies that $\overline{p_i p_j}$ is an edge of $\mathrm{conv}(P)$. For the other direction note that if $\overline{p_i p_j}$ is an edge of $\mathrm{conv}(P)$ then there exists some closed halfplane through p_i and p_j for which $H \cap P = \{p_i, p_j\}$. □

[1]This assumption is here because our definition of convex hull in presence of collinear points is different from what we need here. For three collinear points, we defined the middle one to not be part of the convex hull, whereas here we would like to include it.



7.3 Duality

The *straight-line dual* of a plane graph G is a graph G′ defined as follows: Choose a point in the interior of each face of G and connect any two such points by a straight edge, if the corresponding faces share an edge of G. Observe that this notion depends on the embedding; that is why it is not defined on an abstract graph G but rather on a *plane graph*, a graph embedded in the Euclidean plane such that no two edges intersect, except at common endpoints. In general, G′ may have edge crossings, which may also depend on the choice of representative points within the faces. However, in the following statement there is a particularly natural choice for these points such that G′ is plane: the points from P.

Theorem 7.9 (Delaunay [1]) *The straight-line dual of* $\mathrm{VD}(P)$ *for a set* $P \subset \mathbb{R}^2$ *of* $n \geq 3$ *points in general position (no three points from* P *are collinear and no four points from* P *are cocircular) is a triangulation (→ Delaunay triangulation).*

Proof. By Lemma 7.8, the convex hull edges appear in the straight-line dual T of VD(P). Furthermore, any remaining edge of T is finite, that is, both its endpoints are Voronoi vertices. Consider $v \in \mathrm{VV}(P)$. According to Corollary 7.7b, v is incident to exactly three Voronoi regions, which, therefore, form a triangle $\triangle(v)$ in T. By Corollary 7.7d, $\triangle(v)$ does not contain any point from P in its interior. Hence $\triangle(v)$ appears in the (unique by Theorem 4.10) Delaunay triangulation of P.

Conversely, for any triangle p_i, p_j, p_k in the Delaunay triangulation of P, by the empty circle property the circumcenter c of p_i, p_j, p_k has p_i, p_j, and p_k as its closest points from P. Therefore, $c \in \mathrm{VV}(P)$ and—as above—the triangle p_i, p_j, p_k appears in T. □

Corollary 7.10 $|\mathrm{VE}(P)| \leq 3n - 6$ *and* $|\mathrm{VV}(P)| \leq 2n - 5$.

Proof. Every edge in $VE(P)$ corresponds to an edge in the dual Delaunay triangulation. The latter is a plane graph on n vertices and thus has at most $3n-6$ edges and at most $2n-4$ faces by Lemma 3.1. Only the bounded faces correspond to a vertex in $VD(P)$. □

7.4 Lifting Map

Consider the unit paraboloid $\mathcal{U} : z = x^2 + y^2$ in $\mathbb{R}^3$. The lifting map $u : \mathbb{R}^2 \to \mathcal{U}$ with $u : p = (p_x, p_y, p_z) \mapsto (p_x, p_y, {p_x}^2 + {p_y}^2)$ is the projection of the x/y-plane onto $\mathcal{U}$ in direction of the z-axis.

For $p \in \mathbb{R}^2$ let H_p denote the plane of tangency to $\mathcal{U}$ in $u(p)$. Denote by $h_p : \mathbb{R}^3 \to H_p$ the projection of the x/y-plane onto H_p in direction of the z-axis.

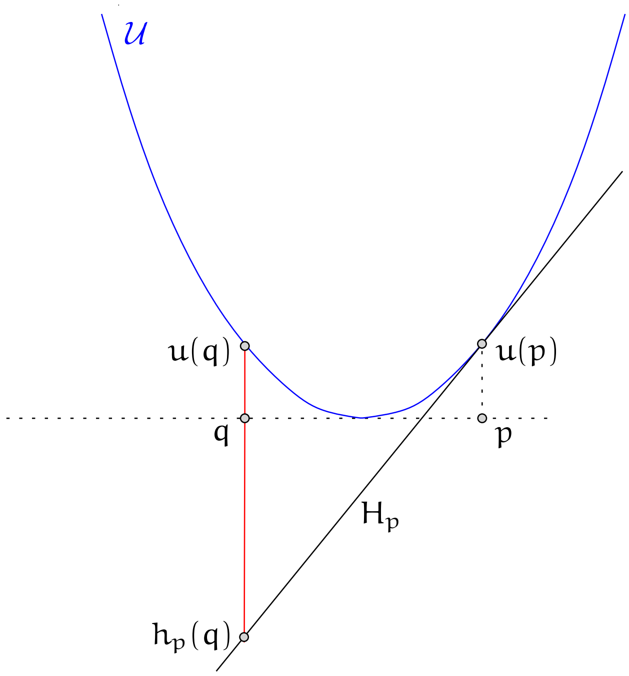


Figure 7.3: *Lifting map* $\mathbb{R}^1 \to \mathbb{R}^2$.

Lemma 7.11 $\|u(q) - h_p(q)\| = \|p - q\|^2$, *for any points* $p, q \in \mathbb{R}^2$.

Proof. Exercise

Theorem 7.12 *Let* $\mathcal{H}(P) := \bigcap_{p \in P} H_p^+$ *the intersection of all halfspaces above the planes* H_p, $p \in P$. *Then the vertical projection of* $\mathcal{H}(P)$ *onto the* x/y*-plane is the Voronoi Diagram of* P.

Proof. For any point $q \in \mathbb{R}^2$, the vertical line through q intersects every plane H_p, $p \in P$. By Lemma 7.11 the topmost plane intersected belongs to the point from P that is closest to q. □

Questions

18. *What is the Voronoi diagram of a set of points in $\mathbb{R}^2$?* Give a precise definition and explain/prove the basic properties: convexity of cells, why is it a subdivision of the plane?, Lemma 7.6, Lemma 7.8.

19. *What is the correspondence between the Voronoi diagram and the Delaunay triangulation for a set of points in $\mathbb{R}^2$?* Prove duality (Theorem 7.9) and explain where general position is needed.

20. *How to construct the Voronoi diagram of a set of points in $\mathbb{R}^2$?* Describe an $O(n \log n)$ time algorithm, for instance, via Delaunay triangulation.

21. *How can the Voronoi diagram be interpreted in context of the lifting map?* Describe the transformation and prove its properties to obtain a formulation of the Voronoi diagram as an intersection of halfspaces one dimension higher.

References

[1] B. Delaunay, Sur la sphère vide. A la memoire de Georges Voronoi, *Izv. Akad. Nauk SSSR, Otdelenie Matematicheskih i Estestvennyh Nauk* **7** (1934), 793–800.