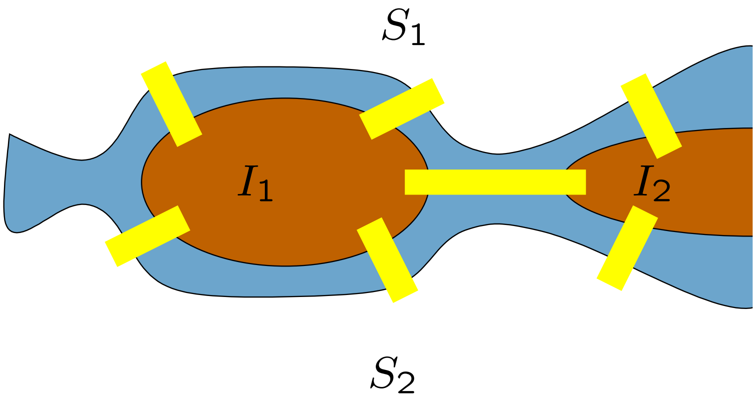
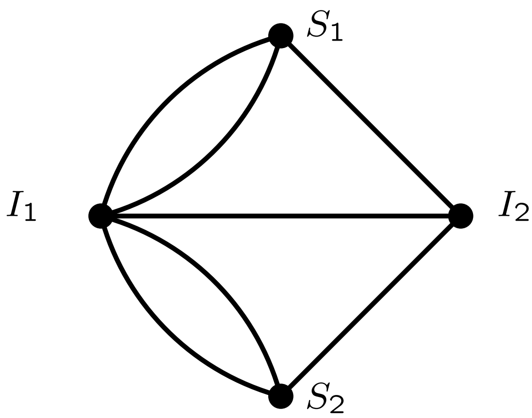
Graph Theory





Graphs – Definition

A graph G is a pair consisting of

- a vertex set $V(G)$, and
- an edge set $E(G) \subseteq \binom{V(G)}{2}$.

x and y are the endpoints of edge $e = \{x, y\}$.
They are called adjacent or neighbors.
e is called incident with x and y.

Multigraphs: Extension & Confusion

A loop is an edge whose endpoints are equal.

Multiple edges are edges having the same set of endpoints.

Our book allows both loops and multiple edges in "graphs". We don't – at least when we say "graph". When we do want to allow multiple edges or loops we say multigraph. When the book wants to talk about a graph without multiple edges and loops, it says simple graph.*

Remarks A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

Every graph is finite.†

*Sometimes even we say "simple graph", when we would like to emphasize that there are no multiple edges and loops.

†in this course

Special graphs

K_n is the complete graph on n vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes n and m.

P_n is the path on n vertices

C_n is the cycle on n vertices

Further definitions

The degree of vertex v is the number of edges incident with v.

A set of pairwise adjacent vertices in a graph is called a clique. A set of pairwise non-adjacent vertices in a graph is called an independent set.

A graph G is bipartite if $V(G)$ is the union of two (possibly empty) independent sets of G. These two sets are called the partite sets of G.

The complement $\overline{G}$ of a graph G is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say G contains H and write $G \supseteq H$.

The Petersen graph

$V(P) = \binom{[5]}{2}$

$E(P) = \{\{A, B\} : A \cap B = \emptyset\}$

Properties.

- each vertex has degree 3 (i.e. P is 3-regular)
- adjacent vertices have no common neighbor
- non-adjacent vertices have exactly one common neighbor

Corollary. The girth of the Petersen graph is 5.

The girth of a graph is the length of its shortest cycle.

Isomorphism of graphs

An isomorphism of G to H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff* $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H, then we say G is isomorphic to H, denoted by $G \cong H$.

Claim. The isomorphism relation is an equivalence relation on the set of all graphs.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation.

Example. What are those graphs for which the adjacency relation is an equivalence relation?

Remark. labeled vs. unlabeled

"unlabeled graph" $\approx$ "isomorphism class".

Example. What is the number of labeled and unlabeled graphs on n vertices?

*if and only if

Equivalence relation

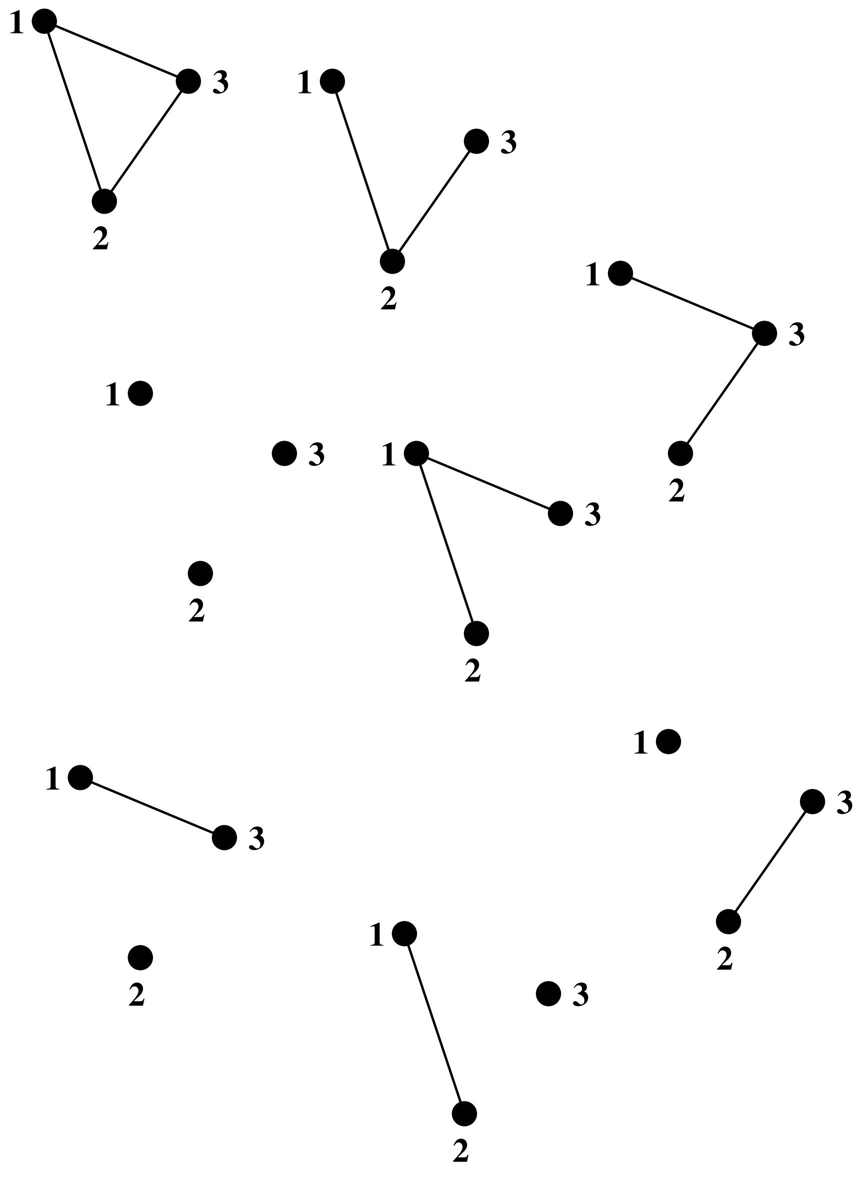
A relation on a set S is a subset of $S \times S$.

A relation R on a set S is an equivalence relation if

1. $(x, x) \in R$ (R is reflexive)
2. $(x, y) \in R$ implies $(y, x) \in R$ (R is symmetric)
3. $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (R is transitive)

An equivalence relation defines a partition of the base set S into equivalence classes. Elements are in relation iff they are within the same class.

Isomorphism classes



Automorphisms

An automorphism of G is an isomorphism of G to G. A graph G is vertex transitive if for every pair of vertices u, v there is an automorphism that maps u to v.

Examples.

- Automorphisms of P_4
- Automorphisms of $K_{r,s}$
- Automorphisms of Petersen graph.

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

A graph is self-complementary if it is isomorphic to its complement.

Example. P_4, C_5

Adjacency matrix of a graph

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix in which entry $a_{i,j}$ is the number of edges whose endpoints are v_i and v_j.

Walks, trails, paths, and cycles

A walk is an alternating list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex.

A u, v-walk, u, v-trail, u, v-path is a walk, trail, path, respectively, with first vertex u and last vertex v.

If $u = v$ then the u, v-walk and u, v-trail is closed. A closed trail (without specifying the first vertex) is a circuit. A circuit with no repeated vertex is called a cycle.

The length of a walk trail, path or cycle is its number of edges.

Connectivity

G is connected, if there is a u, v-path for every pair $u, v \in V(G)$ of vertices.
Otherwise G is disconnected.

Vertex u is connected to vertex v in G if there is a u, v-path. The connection relation on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v-walk contains a u, v-path.

The connected components of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own, called trivial connected component.

Strong Induction

Theorem 1. (Principle of Induction) Let $P(n)$ be a statement with integer parameter n. If the following two conditions hold then $P(n)$ is true for each positive integer n.

1. $P(1)$ is true.

2. For all $n > 1$, "$P(n-1)$ is true" implies "$P(n)$ is true".

Theorem 2. (Strong Principle of Induction) Let $P(n)$ be a statement with integer parameter n. If the following two conditions hold then $P(n)$ is true for each positive integer n.

1. $P(1)$ is true.

2. For all $n > 1$, "$P(k)$ is true for $1 \leq k < n$" implies "$P(n)$ is true".

Cutting a graph

Proposition. Every graph with n vertices and k edges has at least $n-k$ components.

A cut-edge or cut-vertex of G is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G-M$ denotes the graph obtained from G by the deletion of the elements of M; $V(G-M) = V(G)$ and $E(G-M) = E(G) \setminus M$. Similarly, for $S \subseteq V(G)$, $G-S$ obtained from G by the deletion of S and all edges incident with a vertex from S.
For $e \in E(G)$, $G-\{e\}$ is abbreviated by $G-e$.
For $v \in E(G)$, $G-\{v\}$ is abbreviated by $G-v$.

Theorem. An edge e is a cut-edge iff it does not belong to a cycle.