Matchings in general graphs

An odd component is a connected component with an odd number of vertices. Denote by $o(G)$ the number of odd components of a graph G.

Theorem. (Tutte, 1947) A graph G has a perfect matching iff $o(G - S) \leq |S|$ for every subset $S \subseteq V(G)$.

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ (Lovász, 1975) Consider a counterexample G with the maximum number of edges.

Claim. $G + xy$ has a perfect matching for any $xy \notin E(G)$.

Proof of Tutte's Theorem — Continued

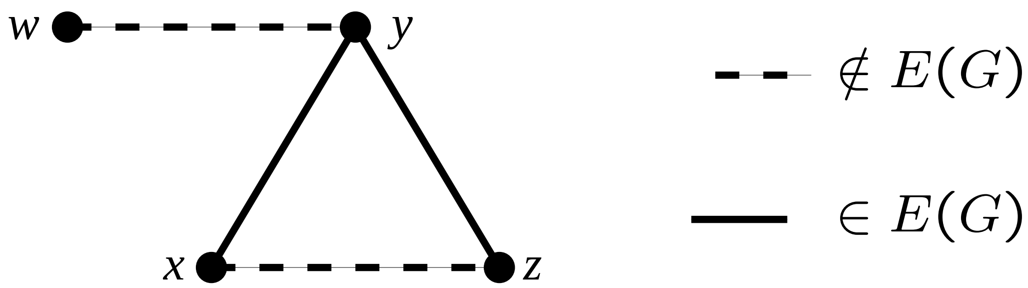
Define $U := \{v \in V(G) : d_G(v) = n(G) - 1\}$

Case 1. $G - U$ consists of disjoint cliques.

Proof: Straightforward to construct a perfect matching of G.

Case 2. $G - U$ is not the disjoint union of cliques.

Proof: Derive the existence of the following subgraph.



Obtain contradiction by constructing a perfect matching M of G using perfect matchings M_1 and M_2 of $G+xz$ and $G + yw$, respectively.

Corollaries

Corollary. (Berge,1958) For a subset $S \subseteq V(G)$ let $d(S) = o(G - S) - |S|$. Then

$$2\alpha'(G) = \min\{n - d(S) : S \subseteq V(G)\}.$$

Proof. $(\leq)$ Easy.

$(\geq)$ Apply Tutte's Theorem to $G \vee K_d$.

Corollary. (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

Proof. Check Tutte's condition. Let $S \subseteq V(G)$.
Double-count the number of edges between an S and the odd components of $G - S$.
Observe that between any odd component and S there are at least three edges.

Factors

A factor of a graph is a spanning subgraph. A k-factor is a spanning k-regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

Theorem. (Petersen, 1891) Every $2k$-regular graph has a 2-factor.

Proof. Use Eulerian cycle of G to create an auxiliary k-regular bipartite graph H, such that a perfect matching in H corresponds to a 2-factor in G.