

## Matchings in general graphs

An **odd component** is a connected component with an odd number of vertices. Denote by  $o(G)$  the number of odd components of a graph  $G$ .

**Theorem.** (Tutte, 1947) A graph  $G$  has a perfect matching iff  $o(G - S) \leq |S|$  for every subset  $S \subseteq V(G)$ .

*Proof.*

$\Rightarrow$  Easy.

$\Leftarrow$  (Lovász, 1975) Consider a counterexample  $G$  with the maximum number of edges.

*Claim.*  $G + xy$  has a perfect matching for any  $xy \notin E(G)$ .

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## Proof of Tutte's Theorem — Continued

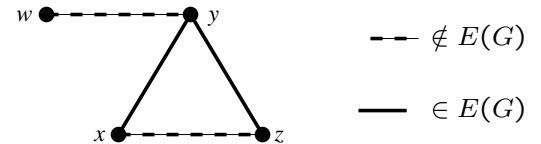
Define  $U := \{v \in V(G) : d_G(v) = n(G) - 1\}$

**Case 1.**  $G - U$  consists of disjoint cliques.

*Proof:* Straightforward to construct a perfect matching of  $G$ .

**Case 2.**  $G - U$  is not the disjoint union of cliques.

*Proof:* Derive the existence of the following subgraph.



Obtain contradiction by constructing a perfect matching  $M$  of  $G$  using perfect matchings  $M_1$  and  $M_2$  of  $G + xz$  and  $G + yw$ , respectively.

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## Corollaries

**Corollary.** (Berge, 1958) For a subset  $S \subseteq V(G)$  let  $d(S) = o(G - S) - |S|$ . Then

$$2\alpha'(G) = \min\{n - d(S) : S \subseteq V(G)\}.$$

*Proof.* ( $\leq$ ) Easy.

( $\geq$ ) Apply Tutte's Theorem to  $G \vee K_d$ .

**Corollary.** (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

*Proof.* Check Tutte's condition. Let  $S \subseteq V(G)$ .

Double-count the number of edges between an  $S$  and the odd components of  $G - S$ .

Observe that between any odd component and  $S$  there are at least three edges.

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## Factors

A **factor** of a graph is a spanning subgraph. A  **$k$ -factor** is a spanning  $k$ -regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

**Theorem.** (Petersen, 1891) Every  $2k$ -regular graph has a 2-factor.

*Proof.* Use Eulerian cycle of  $G$  to create an auxiliary  $k$ -regular bipartite graph  $H$ , such that a perfect matching in  $H$  corresponds to a 2-factor in  $G$ .

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