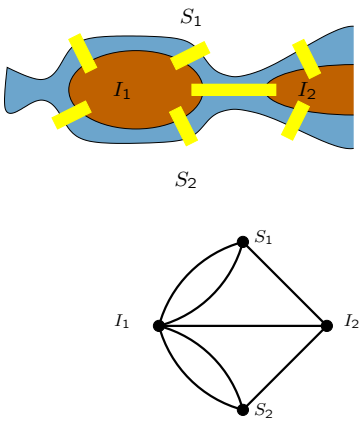
# Graph Theory






## Graphs – Definition

A graph $G$ is a pair consisting of

- a vertex set $V(G)$, and
- an edge set $E(G) \subseteq \binom{V(G)}{2}$.

$x$ and $y$ are the endpoints of edge $e = \{x, y\}$.
They are called adjacent or neighbors.
$e$ is called incident with $x$ and $y$.



## Multigraphs: Extension & Confusion

A loop is an edge whose endpoints are equal.

Multiple edges are edges having the same set of endpoints.

Our book allows both loops and multiple edges in "graphs". We don't – at least when we say "graph". When we do want to allow multiple edges or loops we say multigraph. When the book wants to talk about a graph without multiple edges and loops, it says simple graph.*

**Remarks** A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

Every graph is finite.†

*Sometimes even we say "simple graph", when we would like to emphasize that there are no multiple edges and loops.

†in this course



## Special graphs

$K_n$ is the complete graph on $n$ vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes $n$ and $m$.

$P_n$ is the path on $n$ vertices

$C_n$ is the cycle on $n$ vertices

## Further definitions

The degree of vertex $v$ is the number of edges incident with $v$.

A set of pairwise adjacent vertices in a graph is called a clique. A set of pairwise non-adjacent vertices in a graph is called an independent set.

A graph $G$ is bipartite if $V(G)$ is the union of two (possibly empty) independent sets of $G$. These two sets are called the partite sets of $G$.

The complement $\overline{G}$ of a graph $G$ is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

$H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say $G$ contains $H$ and write $G \supseteq H$.



## The Petersen graph

$V(P) = \binom{[5]}{2}$

$E(P) = \{\{A, B\} : A \cap B = \emptyset\}$

**Properties.**

- each vertex has degree 3 (i.e. $P$ is 3-regular)
- adjacent vertices have no common neighbor
- non-adjacent vertices have exactly one common neighbor

**Corollary.** The girth of the Petersen graph is 5.

The girth of a graph is the length of its shortest cycle.



## Isomorphism of graphs

An isomorphism of $G$ to $H$ is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff* $f(u)f(v) \in E(H)$. If there is an isomorphism from $G$ to $H$, then we say $G$ is isomorphic to $H$, denoted by $G \cong H$.

**Claim.** The isomorphism relation is an equivalence relation on the set of all graphs.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation.

*Example.* What are those graphs for which the adjacency relation is an equivalence relation?

**Remark.** labeled vs. unlabeled

"unlabeled graph" $\approx$ "isomorphism class".

*Example.* What is the number of labeled and unlabeled graphs on $n$ vertices?

*if and only if



## Equivalence relation

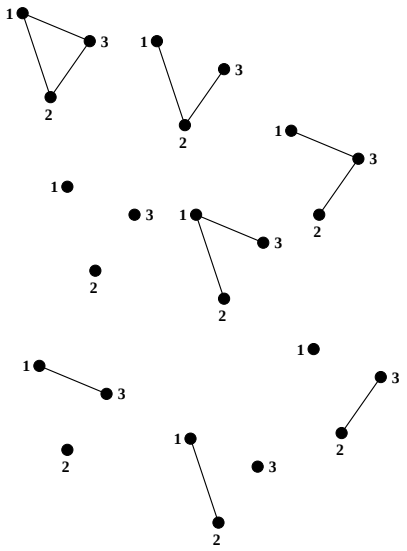
A relation on a set $S$ is a subset of $S \times S$.

A relation $R$ on a set $S$ is an equivalence relation if

1. $(x, x) \in R$ ($R$ is reflexive)
2. $(x, y) \in R$ implies $(y, x) \in R$ ($R$ is symmetric)
3. $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ ($R$ is transitive)

An equivalence relation defines a partition of the base set $S$ into equivalence classes. Elements are in relation iff they are within the same class.

## Isomorphism classes





## Automorphisms

An automorphism of $G$ is an isomorphism of $G$ to $G$. A graph $G$ is vertex transitive if for every pair of vertices $u$, $v$ there is an automorphism that maps $u$ to $v$.

*Examples.*

- Automorphisms of $P_4$
- Automorphisms of $K_{r,s}$
- Automorphisms of Petersen graph.

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

A graph is self-complementary if it is isomorphic to its complement.

*Example.* $P_4, C_5$



## Adjacency matrix of a graph

Let $G$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. The adjacency matrix $A(G)$ of $G$ is an $n \times n$ matrix in which entry $a_{i,j}$ is the number of edges whose endpoints are $v_i$ and $v_j$.



## Walks, trails, paths, and cycles

A walk is an alternating list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex.

A $u, v$-walk, $u, v$-trail, $u, v$-path is a walk, trail, path, respectively, with first vertex $u$ and last vertex $v$.

If $u = v$ then the $u, v$-walk and $u, v$-trail is closed. A closed trail (without specifying the first vertex) is a circuit. A circuit with no repeated vertex is called a cycle.

The length of a walk trail, path or cycle is its number of edges.

## Connectivity

$G$ is connected, if there is a $u, v$-path for every pair $u, v \in V(G)$ of vertices.
Otherwise $G$ is disconnected.

Vertex $u$ is connected to vertex $v$ in $G$ if there is a $u, v$-path. The connection relation on $V(G)$ consists of the ordered pairs $(u, v)$ such that $u$ is connected to $v$.

**Claim.** The connection relation is an equivalence relation.

**Lemma.** Every $u, v$-walk contains a $u, v$-path.

The connected components of $G$ are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own, called trivial connected component.



## Strong Induction

**Theorem 1.** (Principle of Induction) Let $P(n)$ be a statement with integer parameter $n$. If the following two conditions hold then $P(n)$ is true for each positive integer $n$.

1. $P(1)$ is true.
2. For all $n > 1$, "$P(n-1)$ is true" implies "$P(n)$ is true".

**Theorem 2.** (Strong Principle of Induction) Let $P(n)$ be a statement with integer parameter $n$. If the following two conditions hold then $P(n)$ is true for each positive integer $n$.

1. $P(1)$ is true.
2. For all $n > 1$, "$P(k)$ is true for $1 \leq k < n$" implies "$P(n)$ is true".



## Cutting a graph

**Proposition.** Every graph with $n$ vertices and $k$ edges has at least $n - k$ components.

A cut-edge or cut-vertex of $G$ is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G - M$ denotes the graph obtained from $G$ by the deletion of the elements of $M$; $V(G - M) = V(G)$ and $E(G - M) = E(G) \setminus M$. Similarly, for $S \subseteq V(G)$, $G - S$ obtained from $G$ by the deletion of $S$ and all edges incident with a vertex from $S$.
For $e \in E(G)$, $G - \{e\}$ is abbreviated by $G - e$.
For $v \in E(G)$, $G - \{v\}$ is abbreviated by $G - v$.

**Theorem.** An edge $e$ is a cut-edge iff it does not belong to a cycle.