Bipartite graphs

A bipartition of G is a specification of two disjoint independent sets in G whose union is $V(G)$.

Theorem. (König, 1936) A multigraph G is bipartite iff G does not contain an odd cycle.

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ Fix a vertex $v \in V(G)$. Define sets

$$A := \{w \in V(G) : \exists \text{ an odd } v, w\text{-path}\}$$

$$B := \{w \in V(G) : \exists \text{ an even } v, w\text{-path}\}$$

Prove that A and B form a bipartition.

Lemma. Every closed odd walk contains an odd cycle.

Proof. Strong induction.

Eulerian circuits

A multigraph is Eulerian if it has a closed trail containing all its edges. A multigraph is called even if all of its vertices have even degree.

Theorem. Let G be a connected multigraph. Then

$$G \textbf{ is Eulerian iff } G \textbf{ is even.}$$

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ (Strong) induction on the number of edges.

Lemma. If every vertex of a multigraph G has degree at least 2, then G contains a cycle.

Proof. Extremality: Consider a maximal path...

Corollary of the proof. Every even multigraph decomposes into cycles.

Eulerian trails

Theorem. A connected graph with exactly $2k$ vertices of odd degree decomposes into $\max\{k, 1\}$ trails.

Proof. Reduce it to the characterization of Eulerian graphs by introducing auxiliary edges.

Example. The “little house” can be drawn with one continous motion.

Remark. The theorem is “best possible”, i.e. a decomposition into *less* than $\max\{k, 1\}$ trails is not possible.

Proof techniques

- (Strong) induction
- Extremality
- Double counting

Neighborhoods and degrees...

The neighborhood of v in G is
$N_G(v) = \{w \in V(G) : vw \in E(G)\}$.
The degree of a vertex v in graph G is
$d_G(v) = |N_G(v)|$.

The maximum degree of G is $\Delta(G) = \max_{v \in V(G)} d(v)$

The minimum degree of G is $\delta(G) = \min_{v \in V(G)} d(v)$

G is regular if $\Delta(G) = \delta(G)$
G is k-regular if the degree of each vertex is k.

The order of graph G is $n(G) = |V(G)|$.
The size of graph G is $e(G) = |E(G)|$.

Double counting and bijections I

Handshaking Lemma. For any graph G,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Corollary. Every graph has an even number of vertices of odd degree.
No graph of odd order is regular with odd degree.

Corollary. In a graph G the average degree is $\frac{2e(G)}{n(G)}$ and hence $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.

Corollary. A k-regular graph with n vertices has $kn/2$ edges.

The k-dimensional hypercube Q_k

$V(Q_k) = \{0, 1\}^k$

$E(Q_k) = \{xy : x \text{ and } y \text{ differ in exactly one coordinate}\}$

Properties.

- $n(Q_k) = 2^k$
- Q_k is k-regular
- $e(Q_k) = k2^{k-1}$
- Q_k is bipartite
- The number of j-dimensional subcubes (subgraphs isomorphic to Q_j) of Q_k is $\binom{k}{j}2^{k-j}$.

Double counting and bijections II

Proposition. Let G be k-regular bipartite graph with partite sets A and B, $k > 0$. Then $|A| = |B|$.

Proof. Double count the edges of G.

Claim. The Petersen graph contains ten 6-cycles.

Proof. Bijection between 6-cycles and claws. (A claw is a $K_{1,3}$.)