## Directed graphs

A directed (multi)graph (or digraph) is a triple consisting of a vertex set $V(G)$, edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

For an edge $e = (x, y)$, $x$ is the tail of $e$, $y$ is its head.

By path and cycle in a directed graph we always mean directed path and directed cycle.

A directed graph is weakly connected if the underlying undirected graph is connected; it is strongly connected or strong if there is a $u, v$-path for any vertex $u$ and any vertex $v \neq u$.

The out-neighborhood of $v$ in $G$ is
$N_G^+(v) = \{w \in V(G) : (v, w) \in E(G)\}$.
The out-degree of $v$ is $d_G^+(v) = |N_G^+(v)|$.

The in-neighborhood of $v$ in $G$ is
$N_G^-(v) = \{w \in V(G) : (w, v) \in E(G)\}$.
The in-degree of $v$ is $d_G^-(v) = |N_G^-(v)|$.



## Déjà vu

**Directed Handshaking.** In a directed multigraph $G$, we have

$$\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v).$$

A directed multigraph is Eulerian if it has a directed Eulerian circuit, i.e. a closed directed trail containing all edges.

**Theorem.** A weakly connected directed multigraph on $n(D) \geq 2$ vertices is Eulerian iff $d^+(v) = d^-(v)$ for each vertex $v$.

*Proof.* Similar to the undirected case. Think it over.



## Network flows

Network $(D, s, t, c)$; $D$ is a directed multigraph,
$s \in V(D)$ is the source, $t \in V(D)$ is the sink,
$c : E(D) \to \mathbb{R}^+ \cup \{0\}$ is the capacity.

Flow $f$ is a function, $f : E(D) \to \mathbb{R}$

$$f^+(v) := \sum_{v \to u} f(vu)$$

$$f^-(v) := \sum_{u \to v} f(uv).$$

Flow $f$ is feasible if

$(i)$ $f^+(v) = f^-(v)$ for every $v \neq s, t$ (conservation constraints), and

$(ii)$ $0 \leq f(e) \leq c(e)$ for every $e \in E(D)$ (capacity constraints).

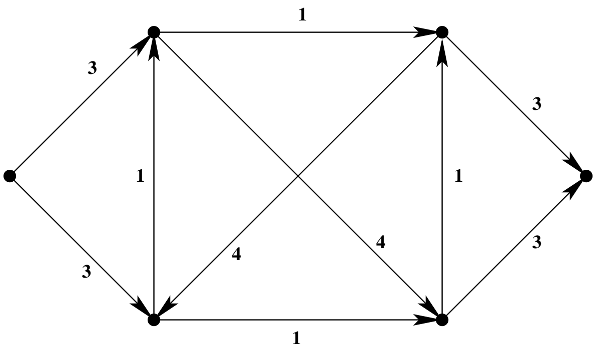
value of flow, $val(f) := f^-(t) - f^+(t)$.

maximum flow: feasible flow with maximum value



## Example

0-flow



## $f$-augmenting path

$G$: underlying undirected graph of network $D$

$s,t$-path $P$ in $G$ is an $f$-augmenting path, if
$s = v_0, e_1, v_1, e_2 \dots v_{k-1}, e_k, v_k = t$ and for every $e_i$

$(i)$ $f(e_i) < c(e_i)$ provided $e_i$ is "forward edge"

$(ii)$ $f(e_i) > 0$ provided $e_i$ is "backward edge"

Tolerance of $P$ is $\min\{\epsilon(e) : e \in E(P)\}$, where
$\epsilon(e) = c(e) - f(e)$ if $e$ is forward, and
$\epsilon(e) = f(e)$ if $e$ is backward.

**Lemma.** Let $f$ be feasible and $P$ be an $f$-augmenting path with tolerance $z$. Define
$f'(e) := f(e) + z$ if $e$ is forward,
$f'(e) := f(e) - z$ if $e$ is backward.
$f'(e) := f(e)$ if $e \notin E(P)$,
Then $f'$ is feasible with $val(f') = val(f) + z$.



## Characterization of maximum flows

**Characterization Lemma.** Feasible flow $f$ is of maximum value iff there is NO $f$-augmenting path.

*Proof.* $\Rightarrow$ Easy.
$\Leftarrow$ Suppose $f$ has no augmenting path.

$S := \{v \in V(D) : \exists\ f$-augmenting path from $s$ to $v\}$.*

Then $t \notin S$ and

$$\sum_{e\in[S,\bar{S}]} c(e) = \sum_{e\in[S,\bar{S}]} f(e) - \sum_{e\in[\bar{S},S]} f(e).$$

We feel, that for any feasible flow $g$ and
any $Q \subseteq V(D), s \in Q, t \notin Q$.

(1) $val(g) \le \sum_{e\in[Q,\bar{Q}]} c(e)$

(2) $val(g) = \sum_{e\in[Q,\bar{Q}]} g(e) - \sum_{e\in[\bar{Q},Q]} g(e)$,

Right? Let's see

*some abuse of definition takes place...



## The value of feasible flow — Proof of (2)

**Lemma.** $g$ any flow, $Q \subseteq V(D)$, then

$$\sum_{e\in[Q,\bar{Q}]} g(e) - \sum_{e\in[\bar{Q},Q]} g(e) = \sum_{v\in Q} (g^+(v) - g^-(v)).$$

In particular, if $g$ is feasible, $s \in Q$, $t \notin Q$, then

$$\sum_{e\in[Q,\bar{Q}]} g(e) - \sum_{e\in[\bar{Q},Q]} g(e) = val(g).$$

*Proof.* For first part: coefficient of $g(e)$ is the same on both sides for every $e \in E(D)$.

For second part:

$$\begin{aligned}\sum_{e\in[\bar{Q},Q]} g(e) - \sum_{e\in[Q,\bar{Q}]} g(e) &= \sum_{v\in\bar{Q}} (g^+(v) - g^-(v)) \\ &= g^+(t) - g^-(t) \\ &= -val(g).\end{aligned}$$

**Remark.** $val(g) = g^+(s) - g^-(s)$.



## Source/sink cuts — Proof of (1)

$[Q,\bar{Q}] = \{(u,v) \in E(D) : u \in Q, v \notin Q\}$ is a source/sink cut, if $s \in Q$ and $t \notin Q$.

capacity of cut: $cap(Q,\bar{Q}) := \sum_{e\in[Q,\bar{Q}]} c(e)$.

**Lemma.** (Weak duality) If $g$ is a feasible flow and $[Q,\bar{Q}]$ is a source/sink cut, then

$$val(f) \le cap(Q,\bar{Q}).$$

*Proof.*

$$\begin{aligned}cap(Q,\bar{Q}) &= \sum_{e\in[Q,\bar{Q}]} c(e) \\ &\ge \sum_{e\in[Q,\bar{Q}]} g(e) \\ &\ge \sum_{e\in[Q,\bar{Q}]} g(e) - \sum_{e\in[\bar{Q},Q]} g(e) \\ &= val(g).\end{aligned}$$

## Max flow-Min cut Theorem

**Max Flow-Min Cut Theorem** (Ford-Fulkerson, 1956)
Let $f$ be a feasible flow of maximum value and $[S, \bar{S}]$ be a source/sink cut of minimum capacity. Then

$$val(f) = cap(S, \bar{S}).$$

*Proof.* (Corollary to proof of Characterization Lemma)
Define

$S := \{v \in V(D) : \exists\ f$-augmenting path from $s$ to $v\}.^*$

Since $f$ is maximum, $f$ ha no augmenting path. Then $t \in \bar{S}$ and of course $s \in S$.

$$\begin{aligned} cap(S, \bar{S}) &= \sum_{e \in [S, \bar{S}]} c(e) \\ &= \sum_{e \in [S, \bar{S}]} f(e) - \sum_{e \in [\bar{S}, S]} f(e) \\ &= val(f). \end{aligned}$$

*some abuse of definition again takes place...



## Ford-Fulkerson Algorithm

**Ford-Fulkerson Algorithm**

**Input.** A feasible flow $f$ in a network $(D, s, t, c)$.

**Output.** EITHER an $f$-augmenting path OR a certificate (a cut with capacity $val(f)$) that $f$ is maximum.

**Idea.** Explore $f$-augmenting paths in the underlying graph $G$ from $s$, letting $R \subseteq V(D)$ the set of vertices reached. Vertices of $R$ that have been explored for path extensions are put in $S$. As a vertex is reached, record the vertex from which it is reached.

**Initialization.** $R = \{s\}$ and $S = \emptyset$.



**Iteration.**

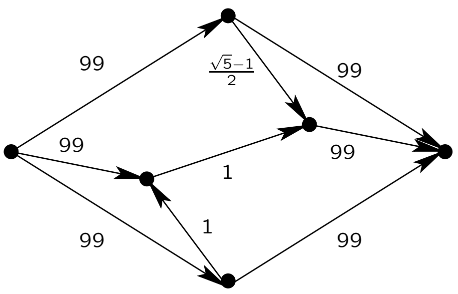
IF $S = R$ THEN
  **stop** and **report** that $f$ is a maximum flow
  and $[S, \bar{S}]$ is a minimum source/sink cut.
ELSE
  select an $x \in R \setminus S$ and explore its neighbors
  $y \in N_G(x)$, for path-extensions.
  IF $xy \in E(D)$ and $f(xy) < c(xy)$ or
    $yx \in E(D)$ and $f(yx) > 0$ THEN
      IF $y = t$ THEN
        **stop** and **report** an $f$-augmenting path.
      ELSE
        **Update** $R := R \cup \{y\}$ ($y$ is reached from $x$),
  After exploring all neighbors of $x$,
  update $S := S \cup \{x\}$, and
**iterate**.

**Theorem.** Repeatedly applying the Ford-Fulkerson Algorithm to a feasible rational flow in a network with rational capacities produces a maximum flow and a minimum source/sink cut.



## Example

The Max-flow Min-cut Theorem is true for real capacities as well,
BUT our algorithm might fail to find a maximum flow!!!




Example of Zwick (1995)

**Remark.** The max flow is 199. There is such an unfortunate choice of a sequence of augmenting paths, by which the flow value tends to 3. Even our implementation can be cheated to do this by introducing an extra vertex in the middle of each edge.

## Integrality Theorem

**Remark.** Edmonds and Karp (1972) modified the FFA to work for real capacities in at most $\frac{n^3-n}{4}$ augmentations.

**Remark.** Our version of the Ford-Fulkerson Algorithm provides a proof of the Max Flow-Min Cut Theorem only for rational capacities. The algorithm is still important for us because it implies the following:

**Corollary.** (**Integrality Theorem**) If all capacities of a network are integers, then there is a maximum flow assigning integral flow to each edge.
Furthermore, some maximum flow can be partitioned into flows of unit value along path from source to sink.



## Directed Edge-Menger

Given $x, y \in V(D)$, a set $F \subseteq E(D)$ is an $x, y$-disconnecting set if $D - F$ has no $x, y$-path. Define

$$\kappa'_D(x,y) := \min\{|F| : F \text{ is an } x,y\text{-disconnecting set,}\}$$
$$\lambda'_D(x,y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* \; x,y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

**Directed-Local-Edge-Menger Theorem** For all $x, y \in V(D)$,

$$\kappa'_D(x,y) = \lambda'_D(x,y).$$

*Proof.* Apply the Integrality Theorem for the network $(D, x, y, c)$ with $c(e) = 1$ for all $e \in E(D)$.

**Corollary** (Directed-Global-Edge-Menger Theorem) Directed multigraph $D$ is strongly $k$-edge-connected iff there is a set of $k$ p.e.d.$x, y$-paths for any two vertices $x$ and $y$.



## Menger's Theorem for directed graphs

Given $x, y \in V(D)$, a set $S \subseteq V(D) \setminus \{x, y\}$ is an $x, y$-separator (or an $x, y$-cut) if $D - S$ has no $x, y$-path.
Define

$$\kappa_D(x,y) := \min\{|S| : S \text{ is an } x,y\text{-cut,}\} \text{ and}$$
$$\lambda_D(x,y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x,y\text{-paths}\}$$

**Directed-Local-Vertex-Menger Theorem** Let $x, y \in V(D)$, such that $xy \notin E(D)$. Then

$$\kappa_D(x,y) = \lambda_D(x,y).$$

*Proof.* We apply the Integrality Theorem for the auxiliary network $(D', x^+, y^-, c')$.

$$V(D') := \{v^-, v^+ : v \in V(D)\}$$
$$E(D') := \{u^+v^- : uv \in E(D)\} \cup \{v^-v^+ : v \in V(D)\}$$
$$c'(u^+v^-) = \infty^* \text{ and } c'(v^-v^+) = 1.$$

*or rather very-very large.



## Corollaries

**Corollary** (Directed-Global-Vertex-Menger Theorem) A digraph $D$ is strongly $k$-connected iff for any two vertices $x, y \in V(D)$ there exist $k$ p.i.d. $x, y$-paths.

*Proof:* **Lemma.** For every $e \in E(D)$, $\kappa_D(G-e) \geq \kappa_D(G)-1$.

And finally, after having 8 versions of Menger's Theorem, the proof of the very first one,
the (original) Undirected-Local-Vertex-Menger Theorem is

**HOMEWORK !!!**

Derive implication DLVM $\Rightarrow$ ULVM