Vertex coloring, chromatic number

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring.

The chromatic number is

$$\chi(G) := \min\{k : \ G \text{ is } k\text{-colorable}\}.$$

A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring.

Examples. K_n, $K_{n,m}$, C_5, Petersen

A graph G is k-color-critical (or k-critical) if $\chi(H) < \chi(G) = k$ for every *proper* subgraph H of G.

Characterization of 1-, 2-, 3-critical graphs.

Lower bounds

Simple lower bounds

$$\begin{aligned} \chi(G) &\geq \omega(G) \\ \chi(G) &\geq \frac{n(G)}{\alpha(G)} \end{aligned}$$

Examples for $\chi(G) \neq \omega(G)$:

- odd cycles of length at least 5,

$$\chi(C_{2k+1}) = 3 > 2 = \omega(C_{2k+1})$$

- complements of odd cycles of order at least 5,

$$\chi(\overline{C}_{2k+1}) = k + 1 > k = \omega(\overline{C}_{2k+1})$$

- random graph $G = G(n, \frac{1}{2})$, almost surely

$$\chi(G) \approx \frac{n}{2 \log n} > 2 \log n \approx \omega(G)$$

Examples for $\chi(G) = \omega(G)$

- cliques, bipartite graphs
- *interval graphs*

  An interval representation of a graph is an assignment of an interval to the vertices of the graph, such that two vertices are adjacent iff the corresponding intervals intersect. A graph having such a representation is called an interval graph.

  Proposition. If G is an interval graph, then

  $$\chi(G) = \omega(G).$$

  Proof. Order vertices according to left endpoints of corresponding intervals and color *greedily*.
- *perfect graphs*

Perfect graphs

Definition (Berge) A graph G is perfect, if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$.

Conjectures of Berge (1960)
Weak Perfect Graph Conjecture. G is perfect iff $\overline{G}$ is perfect.
Strong Perfect Graph Conjecture. G is perfect iff G does not contain an induced subgraph isomorphic to an odd cycle of order at least 5 or the complement of an odd cycle of order at least 5.

The first conjecture was made into the Weak Perfect Graph Theorem by Lovász (1972)
The second conjecture was made into the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, Thomas (2002)

Upper bounds

Proposition $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic; Greedy coloring.

A graph G is d-degenerate if every subgraph of G has minimum degree at most d.

> **Claim.** G is d-degenerate iff there is an ordering of the vertices $v_1, \dots, v_n$, such that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d$

Proposition. For a d-degenerate G, $\chi(G) \leq d + 1$.
In particular, for every G, $\chi(G) \leq \max\limits_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring.

Brooks' Theorem. (1941) Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ iff G is a complete graph or an odd cycle.

Proof. Trickier, but still greedy coloring...
Create ordering $v_1, v_2, \dots, v_n$ for Greedy by a reversed BFS of a rooted spanning tree.
Only v_n (the root) could receive color $\Delta(G) + 1$.

Proof of Brooks' Theorem. Cases.

Case 1. G is not regular.
Let the root be a vertex with degree $< \Delta(G)$.

Case 2. G has a cut-vertex.
Let the root be the cut-vertex.

Assume G is k-regular and $\kappa(G) \geq 2$.

Case 3. $k \leq 2$. Then $G = C_l$ or K_2.

Assume $k \geq 3$. We need a root v_n with nonadjacent neighbors v_1, v_2, such that $G - \{v_1, v_2\}$ is connected.
Let x be a vertex of degree less than $n(G) - 1$.

Case 4. $\kappa(G - x) \geq 2$.
Let v_n be a neighbor of x, which has a neighbor y, such that y and x are non-neighbors. Then let $v_1 = x$ and $v_2 = y$.

Case 5. $\kappa(G - x) = 1$.
Then x has a neighbor in every leaf-block of $G - x$. Let $v_n = x$ and v_1, v_2 be two neighbors of x in different leaf blocks of $G - x$.

Block-decomposition of connected graphs

Maximal induced subgraph of G with no cut-vertex is called block of G.

Lemma. Two blocks intersect in at most one vertex.

Proof. If B_1 and B_2 have no cut-vertex and share at least two vertices then $B_1 \cup B_2$ has no cut-vertex either.

The Block/Cut-vertex graph of G is a bipartite graph with vertex set

$$\{B : B \text{ is a block}\} \cup \{v : v \text{ is a cut-vertex}\}.$$

Block B is adjacent to cut-vertex v iff $v \in V(B)$.

Proposition. The Block/Cut-vertex graph of a connected graph is a tree.

Mycielski's Construction

The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.

Construction. Given graph G with vertices $v_1, \dots, v_n$, we define supergraph $M(G)$.

$$V(M(G)) = V(G) \cup \{u_1, \dots u_n, w\}.$$

$$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}.$$

Theorem.

(i) If G is triangle-free, then so is $M(G)$.

(ii) If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

Forced subdivision

G contains a $K_k \Rightarrow \chi(G) \geq k$

G contains a $K_k \not\Leftarrow \chi(G) \geq k$ (already for $k \geq 3$)

Hajós' Conjecture
G contains a K_k-subdivision $\overset{?}{\Leftarrow} \chi(G) \geq k$

An H-subdivision is a graph obtained from H by successive edge-subdivisions.

Remark. The conjecture is true for $k = 2$ and $k = 3$.

Theorem (Dirac, 1952) Hajós' Conjecture is true for $k = 4$.

Homework. Hajós' Conjecture is false for $k \geq 7$.

Hadwiger's Conjecture
G contains a K_k-*minor* $\overset{?}{\Leftarrow} \chi(G) \geq k$

Proved for $k \leq 6$. Open for $k \geq 7$.

Proof of Dirac's Theorem

Theorem (Dirac, 1952) If $\chi(G) \geq 4$ then G contains a K_4-subdivision.

Proof. Induction on $n(G)$. $n(G) = 4 \Rightarrow G = K_4$.

W.l.o.g. G is 4-critical.

Case 0. $\kappa(G) = 0$ would contradict 4-criticality

Case 1. $\kappa(G) = 1$ would contradict 4-criticality

Case 2. $\kappa(G) = 2$. Let $S = \{x, y\}$ be a cut-set.
$xy \in E(G)$ would contradict 4-criticality

Hence $xy \notin E(G)$.
$\chi(G) \geq 4 \Rightarrow G$ must have an S-lobe H, such that $\chi(H+xy) \geq 4$. Apply induction hypothesis to $H+xy$ and find a K_4-subdivision F in $H + xy$. Then modify F to obtain a K_4-subdivision in G.

Let $S \subseteq V(G)$. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$.

Proof of Dirac's Theorem— Continued

Case 3. $\kappa(G) \geq 3$. Let $x \in V(G)$. $G - x$ is 2-connected, so contains a cycle C of length at least 3.

Claim. There is an x, C-fan of size 3.

Proof. Add a new vertex u to G connecting it to the vertices of C. By the Expansion Lemma the new graph G' is 3-connected. By Menger's Theorem there exist three p.i.d x, u-paths P_1, P_2, P_3 in G'. □

Given a vertex x and a set U of vertices, and x, U-fan is a set of paths from x to U such that any two of them share only the vertex x.
Fan Lemma. G is k-connected iff $|V(G)| \geq k + 1$ and for every choice of $x \in V(G)$ and $U \subseteq V(G)$, $|U| \geq k$, G has an x, U-fan.

Then $C \cup P_1 \cup P_2 \cup P_3 - u$ is K_4-subdivision in G.