Institut für Theoretische Informatik
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Graph Theory — Problem Set 5

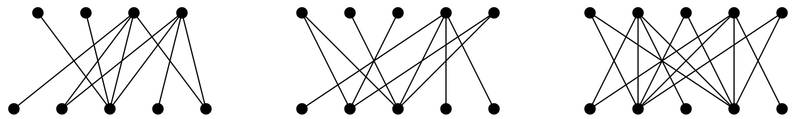
Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: November 24, 2004 at the lecture

Exercise 5.1

(Exercise 3.1.1 in the Textbook)

(−) Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.



Exercise 5.2

(Exercise 3.1.7 in the Textbook)

(−) Prove that a graph G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G with no isolated vertices.

Exercise 5.3

(Exercise 3.1.8 in the Textbook)

(!) Prove or disprove: Every tree has at most one perfect matching.

Exercise 5.4

(Exercise 3.1.9 in the Textbook)

(!) Prove that every **maximal** matching in a graph G has at least $\alpha'(G)/2$ edges.

Exercise 5.5

(Exercise 3.1.24 in the Textbook)

(!) A **permutation matrix** P is a $0,1$-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all row sums and column sums equal k.

Exercise 5.6

(Exercise 3.1.32 in the Textbook)

(!) In an X,Y-bigraph G (namely, a bipartite graph with X and Y as its partite sets), the **deficiency** of a set S is $\mathrm{def}(S) = |S| - |N(S)|$; note that $\mathrm{def}(\emptyset) = 0$. Prove that

$$\alpha'(G) = |X| - \max_{S\subseteq X} \mathrm{def}(S).$$

(Hint: Form a bipartite graph G' such that G' has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.) (Ore [1955])

Exercise 5.7

(Exercise 2.3.14 in the Textbook)

(!) Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C. Prove that there is a minimum-weight spanning tree not containing e. Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.