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Graph Theory — Problem Set 6

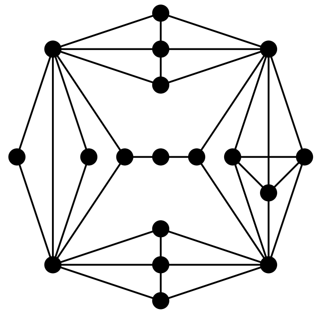
Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: December 1, 2004 at the lecture

Exercise 6.1

(Exercise 3.3.2 in the Textbook)

Exhibit a maximum matching in the graph below, and give a short proof that it has no larger matching.



Exercise 6.2

(Exercise 3.1.18 in the Textbook)

Two people play a game on a graph G, alternately choosing distinct vetices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceeding choice (of the other player). Thus together they follow a path. The last player able to move wins.

Prove that Player 2 has a winning strategy if G has a perfect matching, and otherwise Player 1 has a winning strategy.

Exercise 6.3

(Exercise 3.1.42 in the Textbook)

(!) An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G. (Caro [1979], Wei [1981])

Exercise 6.4

(Exercise 3.3.6 in the Textbook)

(!) Prove that a tree T has a perfect matching if and only if $o(T-v)=1$ for every $v \in V(T)$. (Chungphaisan)

Exercise 6.5

(Exercise 3.3.19 in the Textbook)

(!) Let G be a 3-regular simple graph with no cut-edge. Prove that G decomposes into copies of P_4.

To be continued on the back side...

Exercise 6.6

(Exercise 3.1.25 in the Textbook)

(!) A **doubly stochastic matrix** Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed $Q = c_1 P_1 + \cdots c_m P_m$, where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. (See Exercise 5.5 for the definition of a permutation matrix.) For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(Hint: Use induction on the number of nonzero entries in Q.) (Birkhoff [1946], von Neumann [1953])