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Graph Theory — *Problem Set 8*

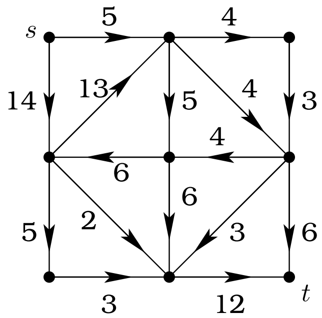
Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: December 15, 2004 at the lecture

Exercise 8.1

(Exercise 4.3.2 in the Textbook)

In the network right below, find a maximum flow from s to t. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.



Exercise 8.2

(Exercise 4.2.12 in the Textbook)

(!) Use Menger's Theorem to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular.

Exercise 8.3

(Exercise 4.2.22 in the Textbook)

(!) Suppose that $\kappa(G) = k$ and $\operatorname{diam} G = d$. Prove that $n(G) \geq k(d-1)+2$ and $\alpha(G) \geq \lceil (1+d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph for which equality holds in both bounds. (The use of Menger's theorem is permitted.)

Exercise 8.4

(Exercise 4.2.23 in the Textbook)

(!) Use Menger's Theorem to prove the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$ when G is bipartite).

To be continued on the back side...

Exercise 8.5

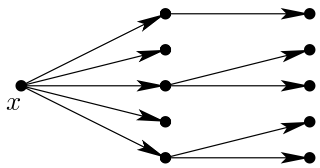
(Exercise scattered in the Textbook)

An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. A **tournament** is an orientation of a complete graph. A **king** in a digraph is a vertex from which every vertex is reachable by a path of length at most 2.

a) (!) Prove that in a tournament every vertex of maximum out-degree is a king.

b) Let D be a tournament having no vertex with in-degree 0. By Part (a), we know that there is a king in a tournament. Prove that if x is a king in D, then D has another king in $N^-(x)$.

c) (+) Let x be a vertex of maximum out-degree in a tournament D. Prove that D has a spanning directed tree rooted at x (i.e., an orientation of a spanning tree where x has in-degree 0) such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity.) (Lu [1996])



Exercise 8.6

(Exercise not in the Textbook)

In the eighth grade, playing in the little soccer league, our school (*Áldás*) was in a fierce competition with four others: the *Ady*, the *Fillér*, the *Medve*, and the *Törökvészi* elementary schools. During the year every team played every other team 6 times. Despite my brilliant effort as right mid-fielder, in the middle of the season we were standing without a single win, with twelve losses. Our coach was still very enthusiastic and in order to motivate us, got into some complicated argument about how we can still win at least a share of the championship (i.e. no other school will have more wins then us at the end).

The situation was the following. Ady had 5 more games with Fillér, 2 more games with Medve, and 5 with Törökvészi. Fillér had 3 more games with Medve, and 6 with Törökvészi. Medve had to play 3 more times with Törökvészi. Ady was at first place with 8 wins, Törökvészi was second with 7, Fillér had 6, and even the much despised Medve was ahead of us with 3 wins.

Was our coach right about his calculations or did he just want to fire us up before our usual showdown with Medve? (Hint: you can try to model the problem as a network flow.)