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Graph Theory — *Problem Set 1*

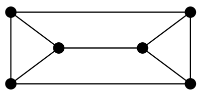
Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: October 27, 2004 at the lecture

Exercise 1.1

(Exercise 1.1.6 in the Textbook)

(–) A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. Determine whether the graph below decomposes into copies of P_4.



Exercise 1.2

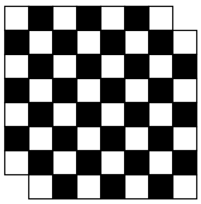
(Exercise 1.1.7 in the Textbook)

(–) Prove that a graph with more than six vertices of odd degree cannot be decomposed into three paths.

Exercise 1.3

(Exercise 1.1.14 in the Textbook)

(!) Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a sub-board that cannot be partitioned into 1-by-2 and 2-by-1 rectangles.

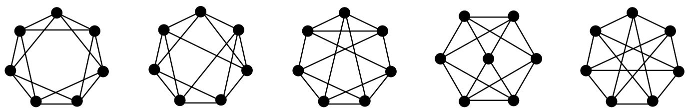


Using the same argument, make a general statement about all bipartite graphs.

Exercise 1.4

(Exercise 1.1.22 in the Textbook)

(!) Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.



Exercise 1.5

(Exercise 1.1.27 in the Textbook)

(!) Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least k^2+1 vertices. For $k=2$ and $k=3$, find one such graph with exactly k^2+1 vertices.

Exercise 1.6

(Exercise 1.1.29 in the Textbook)

Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers. Solve by phrasing the problem in a graph-theoretic way.

Exercise 1.7

(Exercise 1.1.31 in the Textbook)

(!) A graph is **self-complementary** if it is isomorphic to its complement. Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4. (Hint: When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1 \bmod 4$, add one vertex to the graph constructed for $n-1$.)

Exercise 1.8

(Exercise 1.1.47 in the Textbook)

($*$) *Edge-transitive versus vertex-transitive.* An **automorphism** of a graph G is an isomorphism of G to G. A graph G is **vertex-transitive** if for every pair of vertices u, v there is an automorphism that maps u to v. A graph G is **edge-transitive** if for every pair of edges e, f there is an automorphism that maps the endpoints of e to the endpoints of f (in either order).

a) Let G be obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.

b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.

c) Prove that the graph in Class Exercise 1.1 is vertex-transitive but not edge-transitive.

Exercise 1.9

(Exercise 1.1.28 in the Textbook)

(+) *The Odd Graph* O_k. The vertices of the graph O_k are the k-element subsets of $\{1, 2, \ldots, 2k+1\}$. Two vertices are adjacent if and only if they are disjoint sets. Thus O_2 is the Petersen graph. Prove that the girth of O_k is 6 if $k \geq 3$.