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Graph Theory — Problem Set 4

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: November 17, 2004 at the lecture

Exercise 4.1

(Exercise 2.1.8 in the Textbook)

(−) Prove that each property below characterizes forests.

a) Every induced subgraph has a vertex of degree at most 1.
b) Every connected subgraph is an induced subgraph.
c) The number of components is the number of vertices minus the number of edges.

An **induced subgraph** is a subgraph obtained by deleting a set of vertices. We write $G[U]$ for $G - \overline{U}$, where $\overline{U} = V(G) - U$; this is the subgraph of G induced by U. In other words, $G[U]$ is a subgraph of G with vertex set U which contains all the edges $xy \in E(G)$ with $x, y \in U$.

Exercise 4.2

(Exercise 2.1.27 in the Textbook)

(!) Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^n d_i = 2n - 2$.

Exercise 4.3

(Exercise 2.1.37 in the Textbook)

(!) Let T, T' be two spanning trees of a connected graph G. For any $e \in E(T) - E(T')$, prove that there exists an edge $e' \in E(T') - E(T)$ such that both $T' + e - e'$ and $T - e + e'$ become spanning trees of G simultaneously.

Exercise 4.4

(Exercise 2.1.47 in the Textbook)

(!) If a graph G has a u, v-path, then the **distance** from u to v, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the least length of a u, v-path. (Remember the length of path is the number of its edges.) If G has no such path, then $d(u, v) = \infty$. The **diameter** of G is $\max_{u,v \in V(G)} d(u, v)$ and denoted by $\operatorname{diam} G$. The **eccentricity** of a vertex u is $\max_{v \in V(G)} d(u, v)$, and denoted by $\epsilon(u)$. The **radius** of a graph G is $\min_{u \in V(G)} \epsilon(u)$, and denoted by $\operatorname{rad} G$.

a) Prove that the distance function $d(u, v)$ on pairs of vertices of graph satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.
b) Use part (a) to prove that $\operatorname{diam} G \leq 2 \operatorname{rad} G$ for every graph G.
c) For all positive integers r and d that satisfy $r \leq d \leq 2r$, construct a simple graph with radius r and diameter d. (Hint: Build a suitable graph with one cycle.)

Exercise 4.5

(Exercise 2.3.14 in the Textbook)

(!) Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C. Prove that there is a minimum-weight spanning tree not containing e. Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.

To be continued on the back side...

Exercise 4.6

(Exercise not in the Textbook)

(+) Let k be a positive integer.

a) Prove that every simple n-vertex graph with more than $n(k-1) - \binom{k}{2}$ edges contains **all** trees with k edges, if $n > k$. (Hint: Try to prove and use the following claim: If $\delta(G) \geq k$ then G contains all trees with k edges.)

b) For every k, construct a simple n-vertex graph, for some $n > k$, with $n(k-1)/2$ edges which contains **no** tree with k edges.

c) For $k \in \{1, 2, 3\}$, prove that every simple n-vertex graph with more than $n(k-1)/2$ edges contains **all** trees with k edges.