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Graph Theory — *Problem Set 9*

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: December 22, 2004 at the lecture

Exercise 9.1

(Exercise 4.2.21 in the Textbook)

(!) Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Nash-Williams [1960])

Exercise 9.2

(Exercise 4.2.29 in the Textbook)

Given a graph G, let D be the digraph obtained by replacing each edge with two oppositely-directed edges having the same endpoints (thus D is the symmetric digraph with underlying graph G). Assume that, for all $x, y \in V(D)$, $\kappa_D(x,y) = \lambda_D(x,y)$ holds when $(x,y) \notin E(D)$. Use this hypothesis to prove that also $\kappa_G(x,y) = \lambda_G(x,y)$ for $\{x,y\} \notin E(G)$.

Exercise 9.3

(Exercise not in the Textbook)

In the eighth grade, playing in the little soccer league, our school (*Áldás*) was in a fierce competition with four others: the *Ady*, the *Fillér*, the *Medve*, and the *Törökvészi* elementary schools. During the year every team played every other team 6 times. Despite my brilliant effort as right mid-fielder, in the middle of the season we were standing without a single win, with twelve losses. Our coach was still very enthusiastic and in order to motivate us, got into some complicated argument about how we can still win at least a share of the championship.

The situation was the following. Ady had 5 more games with Fillér, 2 more games with Medve, and 5 with Törökvészi. Fillér had 3 more games with Medve, and 6 with Törökvészi. Medve had to play 3 more times with Törökvészi. Ady was at first place with 8 wins, Törökvészi was second with 7, Fillér had 6, and even the much despised Medve was ahead of us with 3 wins.

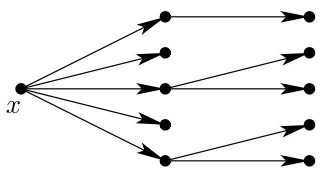
Was our coach right about his calculations or did he just want to fire us up before our usual showdown with Medve? (Hint: you can try to model the problem as a network flow.)

Exercise 9.4

(Exercise scattered in the Textbook)

An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. A **tournament** is an orientation of a complete graph. A **king** in a digraph is a vertex from which every vertex is reachable by a path of length at most 2.

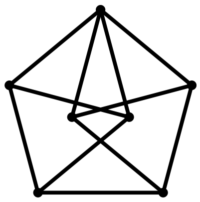
c) (+) Let x be a vertex of maximum out-degree in a tournament D. Prove that D has a spanning directed tree rooted at x (i.e., an orientation of a spanning tree where x has in-degree 0) such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity.) (Lu [1996])



Exercise 9.5

(Exercise 5.1.1 in the Textbook)

(−) Compute the clique number, the independence number and the chromatic number of the graph below. Is the graph color-critical?



Exercise 9.6

(Exercise 5.1.20 in the Textbook)

(!) Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

Exercise 9.7

(Exercise 5.1.41 in the Textbook)

(!) Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Nordhaus–Gaddum [1956])