Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Dr. Tibor Szabó and Miloš Stojaković

February 2, 2005

Graph Theory — *Problem Set 14*

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT04/`

Due Date: February 9, 2005

Exercise 14.1

(Exercise 6.3.16 in the Textbook)

($-$) Prove that the 4-dimensional cube Q_4 is nonplanar. Decompose it into two isomorphic planar graphs.

Exercise 14.2

(Exercise not in the Textbook)

Prove that every outerplanar graph is 3-colorable following each of the ideas below.

a) Use Four Color Theorem.

b) Use Dirac's theorem.

c) Show every simple outerplanar graph is 2-degenerate.

Exercise 14.3

(Exercise not in the Textbook)

In a graph G, to **contract** an edge e with endpoints u, v is to replace u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v. (Hence, the resulting graph has one less edge than G.) A graph H is a **minor** of G if a copy of H can be obtained from G by deleting and/or contracting edges of G, and/or by deleting vertices of G. In such a case we say that **G contains an H-minor**.

Show that the Petersen graph contains a K_5-minor and a $K_{3,3}$-minor.

Exercise 14.4

(Exercise 6.2.12 in the Textbook)

(!) Wagner [1937] proved that the following condition is necessary and sufficient for a graph G to be planar: G contains neither K_5-minor nor $K_{3,3}$-minor.

a) Show that deletion and contraction of edges preserve planarity. Conclude from this that Wagner's condition is necessary.

b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.

Exercise 14.5

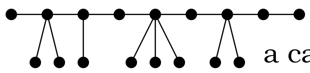
(Exercise 6.3.14 in the Textbook)

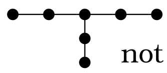
(+) Prove that a maximal planar graph is 3-colorable if and only if it is Eulerian. (Hint: For sufficiency, use induction on $n(G)$. Choose an appropriate pair or triple of adjacent vertices to replace with appropriate edges.) (Heawood [1898])

Exercise 14.6

(Exercise not in the Textbook)

A **caterpillar** is a tree in which a single path, called the **spine**, is incident to (or contains) every edge. Prove that in every infinite family of caterpillars there exists two such that one is a minor of the other. (This proves Graph Minor Theorem for caterpillars.)


a caterpillar


not a caterpillar