Line graphs and edge coloring

A $k$-edge-coloring of a multigraph $G$ is a labeling $f : E(G) \to S$, where $|S| = k$. The labels are called colors; the edges of one color form a color class. A $k$-edge-coloring is proper if incident edges have different labels. A multigraph is $k$-edge-colorable if it has a proper $k$-edge-coloring.

The edge-chromatic number (or chromatic index) of a loopless multigraph $G$ is

$$\chi'(G) := \min\{k : G \text{ is } k\text{-edge-colorable}\}.$$  

A multigraph $G$ is $k$-edge-chromatic if $\chi'(G) = k$.

Remarks. $\chi'(G) = \chi(L(G))$, so

$$\Delta(G) \leq \omega(L(G)) \leq \chi'(G) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1.$$  

Vizing’s Theorem

Example. $K_{2n}$

Theorem. (König, 1916) For a bipartite multigraph $G$, $\chi'(G) = \Delta(G)$

Proposition. $\chi'(Petersen) = 4$.

Theorem. (Vizing, 1964) For a simple graph $G$,

$$\chi'(G) \leq \Delta(G) + 1.$$  

Generalization. If the maximum edge-multiplicity in a multigraph $G$ is $\mu(G)$, then $\chi'(G) \leq \Delta(G) + \mu(G)$

Example. Fat triangle; $\chi'(G) = \Delta(G) + \mu(G)$.

Proof of Vizing’s Theorem (A. Schrijver)

Induction on $n(G)$.

If $n(G) = 1$, then $G = K_1$; the theorem is OK.

Assume $n(G) > 1$. Delete a vertex $v \in V(G)$. By induction $G - v$ is $(\Delta(G) + 1)$-edge-colorable.

Why is $G$ also $(\Delta(G) + 1)$-edge-colorable?

We prove the following

Stronger Statement. Let $k \geq 1$ be an integer. Let $v \in V(G)$, such that

- $d(v) \leq k$,
- $d(u) \leq k$ for every $u \in N(v)$, and
- $d(u) = k$ for at most one $u \in N(v)$.

Then $G - v$ is $k$-edge-colorable $\Rightarrow$ $G$ is $k$-edge-colorable.

Proof of the Stronger Statement I

Induction $k$ (!!!)

For $k = 1$ it is OK.

W.l.o.g. $d(u) = k - 1$ for every $u \in N(v)$, except for exactly one $w \in N(v)$ for which $d(w) = k$.

Let $f : E(G) \to \{1, \ldots, k\}$ be a proper $k$-edge-coloring of $G - v$, which minimizes

$$\sum_{i=1}^{k} |X_i|^2.$$  

Here $X_i := \{ u \in N(v) : u \text{ is missing color } i \}$. 

Proof of the Stronger Statement II

Case 1. There is an \( i \), with \( \mathcal{X}_{ij} = 1 \). Say \( X_k = \{u\} \).

Let \( G' = G - uv - \{xy : f(xy) = k\} \).

Apply the induction hypothesis for \( G' \) and \( k - 1 \).

Case 2. \( |X_i| \neq 1 \) for every \( i = 1, \ldots, k \).

Then
\[
\sum_{i=1}^{k} |X_i| = 2d(v) - 1 < 2k.
\]

So there are colors \( i \) with \( |X_i| = 0 \) and \( j \) with an odd \( |X_j| \neq 1 \).

\( H \) is subgraph spanned by edges of color \( i \) and \( j \).

Switch colors \( i \) and \( j \) in a component \( C \) of \( H \), where \( |C \cap X_j| = 1 \).

This reduces \( \sum_{i=1}^{k} |X_i|^2 \), a contradiction.