Graph Theory — Old Exams

Exam — March 4th, 2003

1. (12 points) Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2. (12 points) Determine the maximum number of edges in a simple bipartite graph that contains no matching with 10 edges and no star with 8 edges.

3. (12 points) (a) Determine the number of triangles in the line graph of $Q_5$.

   (b) Prove that the Petersen graph is non-planar.

4. (12 points) Prim’s Algorithm grows a spanning tree from a given vertex of a connected weighted graph $G$, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of $G$ have been reached. (Ties are broken arbitrarily.) Prove that Prim’s algorithm produces a minimum-weight spanning tree of $G$.

5. (12 points) Prove that a tree $T$ has a perfect matching if and only if $\omega(T - v) = 1$ for every $v \in V(T)$.

6. (18 points) (a) Prove that no matter how you color the edges of a $K_{10}$ with RED/BLUE, you will find either a completely RED $K_3$ or a completely BLUE $K_4$. (Hint: You can apply the fact, that no matter how you color the edges of $K_6$ with RED/BLUE there will be a monochromatic triangle; We formulated and proved this statement in terms of acquaintances and non-acquaintances among six people.)

   (b) Prove the same statement with $K_9$ instead of $K_{10}$! (Hint: Look at more carefully how a counterexample would look like.)

   (c) Show that the same statement is not true for $K_8$.

Exam — March 2nd, 2004

1. (Not covered in this year’s 2006/07 course!)

   Find a maximum flow from $s$ to $t$ in the network below. Give a short proof that your flow is optimal. (The numbers on edges represent the capacities.)

2. Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^n d_i = 2n - 2$. 
3. Let $G$ be a simple bipartite graph. Prove that the complement of the line graph of $G$ is perfect. (A solution using the weak perfect graph theorem will receive only half of the credit.)

4. Let $G$ be an $(r - 1)$-edge-connected $r$-regular simple graph with even number of vertices, where $r$ is a positive integer. Prove that $G$ has a perfect matching.

5. Denote by $G$ the class of bipartite graphs with 161 edges and maximum degree 7. Determine the minimum of the maximum matching size within the class $G$, namely, $\min_{G \in G} \alpha'(G)$.

6. Prove the Four Color Theorem for triangle-free planar graphs.

Exam — March 1st, 2005

1. Let $T$ and $T'$ be two spanning trees of a connected graph $G$. For any $e \in E(T) \setminus E(T')$, prove that there exists an edge $e' \in E(T') \setminus E(T)$, such that both $T' + e - e'$ and $T - e + e'$ are spanning trees of $G$.

2. Let $G$ be a 3-regular graph. Prove that $G$ has a vertex two-coloring, such that each monochromatic component has at most two vertices.

3. Somebody distributed a deck of cards into 13 packs of 4 cards. Show that you can select one of each kind by selecting one from each pack. (A deck contains 52 cards. Each card has a “suit” and a “kind”. There are four suits: spades, heart, diamond, and clubs. There are 13 kinds: 2,3,4,5,6,7,8,9,10, J, Q, K, A.)

4. The Grötsch graph is created from $C_5$ by applying Myczyński’s construction. Construct a graph for every $n$ which has no subgraph isomorphic to the Grötsch graph and contains $\lceil \frac{n^2}{4} \rceil$ edges.

5. Let $G$ be a simple planar graph with at least four vertices. Prove that $G$ has at least four vertices whose degree is less than 6.

6. Let $G \neq K_2$ be a connected, regular bipartite graph. Show that $G$ is 2-connected.

Exam — March 7th, 2006

1. Prove or disprove: Every 4-regular planar graph has a triangle.

2. Prove that $X_1(G) + X_1(\overline{G}) \leq n + 1$.

3. Let $G$ be an $r$-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that $G$ has a perfect matching.

4. Let $k \leq n - 1$. Prove that if $\delta(G) \geq \frac{n+k-2}{2}$, then $\kappa(G) \geq k$. 

5. Let $G$ be a simple bipartite graph. Prove that the complement of the line graph of $G$ is perfect. (A solution using the weak perfect graph theorem will receive only half of the credit.)

6. Let $C$ be a cycle in a connected weighted graph. Let $e$ be an edge of maximum weight on $C$. Prove that there is a minimum-weight spanning tree not containing $e$.
Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.