Δ' than to Δ (Figure 5.10c). But this is a contradiction to our choice of Δ and p. Hence there was no (Δ, p), and $\mathcal{D}$ is a Delaunay triangulation. □

Exercise 5.14 *The Euclidean minimum spanning tree (EMST) of a finite point set* $P \subset \mathbb{R}^2$ *is a spanning tree for which the sum of the edge lengths is minimum (among all spanning trees of* P*). Show:*

a) Every EMST of P *is a plane graph.*

b) Every EMST of P *contains a closest pair, i.e., an edge between two points* $p, q \in P$ *that have minimum distance to each other among all point pairs in* $\binom{P}{2}$.

c) Every Delaunay Triangulation of P *contains an EMST of* P.

5.5 The Delaunay Graph

Despite the fact that a point set may have more than one Delaunay triangulation, there are certain edges that are present in every Delaunay triangulation, for instance, the edges of the convex hull.

Definition 5.15 *The* **Delaunay graph** *of* $P \subseteq \mathbb{R}^2$ *consists of all line segments* $\overline{pq}$*, for* $p, q \in P$*, that are contained in every Delaunay triangulation of* P.

The following characterizes the edges of the Delaunay graph.

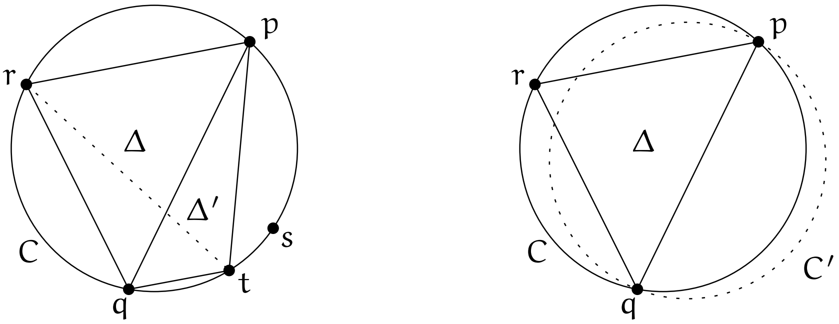
Lemma 5.16 *The segment* $\overline{pq}$*, for* $p, q \in P$*, is in the Delaunay graph of* P *if and only if there exists a circle through* p *and* q *that has* p *and* q *on its boundary and all other points of* P *are strictly outside.*

Proof. "⇒": Let pq be an edge in the Delaunay graph of P, and let $\mathcal{D}$ be a Delaunay triangulation of P. Then there exists a triangle $\Delta = pqr$ in $\mathcal{D}$, whose circumcircle C does not contain any point from P in its interior.

If there is a point s on ∂C such that $\overline{rs}$ intersects $\overline{pq}$, then let $\Delta' = pqt$ denote the other ($\neq \Delta$) triangle in $\mathcal{D}$ that is incident to pq (Figure 5.11a). Flipping the edge pq to rt yields another Delaunay triangulation of P that does not contain the edge pq, in contradiction to pq being an edge in the Delaunay graph of P. Therefore, there is no such point s.

Otherwise we can slightly change the circle C by moving away from r while keeping p and q on the circle. As P is a finite point set, we can do such a modification without catching another point from P with the circle. In this way we obtain a circle C' through p and q such that all other points from P are strictly outside C' (Figure 5.12b).

"⇐": Let $\mathcal{D}$ be a Delaunay triangulation of P. If $\overline{pq}$ is not an edge of $\mathcal{D}$, there must be another edge of $\mathcal{D}$ that crosses $\overline{pq}$ (otherwise, we could add $\overline{pq}$ to $\mathcal{D}$ and still have



(a) Another point $s \in \partial C$. **(b)** Moving C away from s.

Figure 5.11: *Characterization of edges in the Delaunay graph (I).*

a plane graph, a contradiction to $\mathcal{D}$ being a triangulation of P). Let rs denote the first edge of $\mathcal{D}$ that intersects the directed line segment $\overline{pq}$.

Consider the triangle Δ of $\mathcal{D}$ that is incident to rs on the side that faces p (given that $\overline{rs}$ intersects $\overline{pq}$ this is a well defined direction). By the choice of rs neither of the other two edges of Δ intersects $\overline{pq}$, and $p \notin \Delta^\circ$ because Δ is part of a triangulation of P. The only remaining option is that p is a vertex of $\Delta = prs$. As Δ is part of a Delaunay triangulation, its circumcircle C_Δ is empty (i.e., ${C_\Delta}^\circ \cap P = \emptyset$).

Consider now a circle C through p and q, which exists by assumption. Fixing p and q, expand C towards r to eventually obtain the circle C' through p, q, and r (Figure 5.12a). Recall that r and s are on different sides of the line through p and q. Therefore, s lies strictly outside of C'. Next fix p and r and expand C' towards s to eventually obtain the circle C_Δ through p, r, and s (Figure 5.12b). Recall that s and q are on the same side of the line through p and r. Therefore, $q \in C_\Delta$, which is in contradiction to C_Δ being empty. It follows that there is no Delaunay triangulation of P that does not contain the edge pq. □

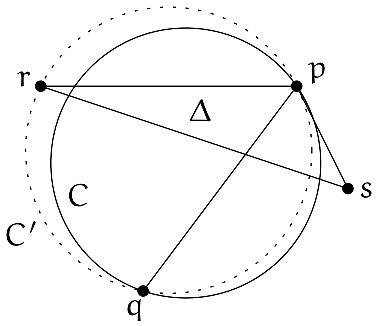
The Delaunay graph is useful to prove uniqueness of the Delaunay triangulation in case of general position.

Corollary 5.17 *Let $P \subset \mathbb{R}^2$ be a finite set of points in general position, that is, no four points of P are cocircular. Then P has a unique Delaunay triangulation.* □

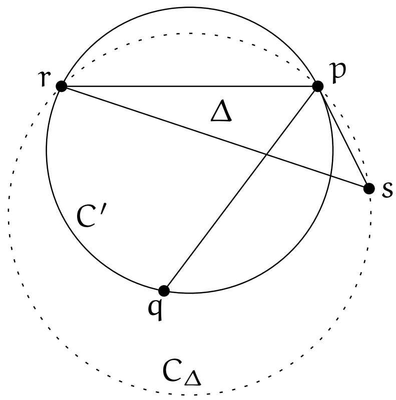
5.6 Every Delaunay Triangulation Maximizes the Smallest Angle

Why are we actually interested in Delaunay triangulations? After all, having empty circumcircles is not a goal in itself. But it turns out that Delaunay triangulations satisfy a number of interesting properties. Here we show just one of them.

Recall that when we compared a scan triangulation with a Delaunay triangulation of the same point set in Figure 5.3, we claimed that the scan triangulation is “ugly” because it contains many long and skinny triangles. The triangles of the Delaunay triangulation,



(a) Expanding C towards r.



(b) Expanding C′ towards s.

Figure 5.12: *Characterization of edges in the Delaunay graph (II).*

at least in this example, look much nicer, that is, much closer to an equilateral triangle. One way to quantify this "niceness" is to look at the angles that appear in a triangulation: If all angles are large, then all triangles are reasonably close to an equilateral triangle. Indeed, we will show that Delaunay triangulations maximize the smallest angle among all triangulations of a given point set. Note that this does not imply that there are no long and skinny triangles in a Delaunay triangulation. But if there is a long and skinny triangle in a Delaunay triangulation, then there is an at least as long and skinny triangle in *every* triangulation of the point set.

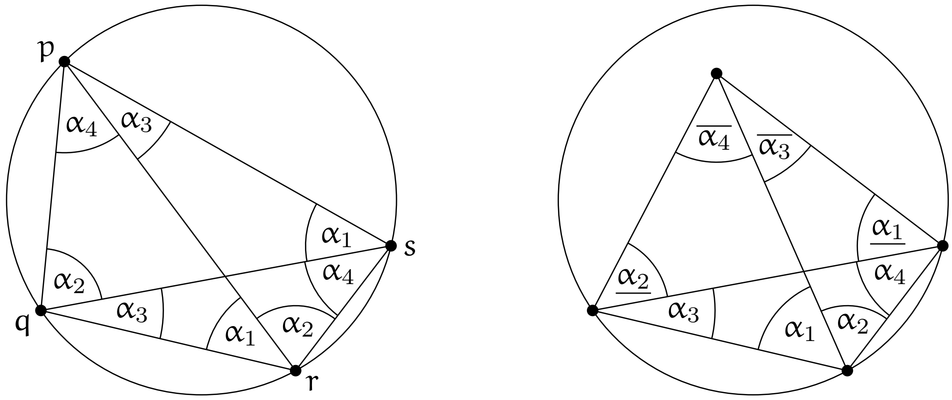
Given a triangulation $\mathcal{T}$ of P, consider the sorted sequence $A(\mathcal{T}) = (\alpha_1, \alpha_2, \ldots, \alpha_{3m})$ of interior angles, where m is the number of triangles (we have already remarked earlier that m is a function of P only and does not depend on $\mathcal{T}$). Being sorted means that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{3m}$. Let $\mathcal{T}, \mathcal{T}'$ be two triangulations of P. We say that $A(\mathcal{T}) < A(\mathcal{T}')$ if there exists some i for which $\alpha_i < \alpha'_i$ and $\alpha_j = \alpha'_j$, for all $j < i$. (This is nothing but the lexicographic order on these sequences.)

Theorem 5.18 *Let $P \subseteq \mathbb{R}^2$ be a finite set of points in general position (not all collinear and no four cocircular). Let $\mathcal{D}^*$ be the unique Delaunay triangulation of P, and let $\mathcal{T}$ be any triangulation of P. Then $A(\mathcal{T}) \leqslant A(\mathcal{D}^*)$.*

In particular, $\mathcal{D}^*$ maximizes the smallest angle among all triangulations of P.

Proof. We know that $\mathcal{T}$ can be transformed into $\mathcal{D}^*$ through the Lawson flip algorithm, and we are done if we can show that each such flip lexicographically increases the sorted angle sequence. A flip replaces six interior angles by six other interior angles, and we will actually show that the smallest of the six angles *strictly* increases under the flip. This implies that the whole angle sequence increases lexicographically.

Let us first look at the situation of four cocircular points, see Figure 5.13a. In this situation, the *inscribed angle theorem* (a generalization of Thales' Theorem, stated below as Theorem 5.19) tells us that the eight depicted angles come in four equal pairs.



(a) Four cocircular points and the induced eight angles.

(b) The situation before a flip.

Figure 5.13: *Angle-optimality of Delaunay triangulations.*

For instance, the angles labeled α_1 at s and r are angles on the same side of the chord pq of the circle.

In Figure 5.13b, we have the situation in which we perform a Lawson flip (replacing the solid with the dashed diagonal). By the symbol $\underline{\alpha}$ ($\overline{\alpha}$, respectively) we denote an angle strictly smaller (larger, respectively) than α. Here are the six angles before the flip:

$$\alpha_1 + \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \underline{\alpha_1}, \quad \underline{\alpha_2}, \quad \overline{\alpha_3} + \overline{\alpha_4}.$$

After the flip, we have

$$\alpha_1, \quad \alpha_2, \quad \overline{\alpha_3}, \quad \overline{\alpha_4}, \quad \underline{\alpha_1} + \alpha_4, \quad \underline{\alpha_2} + \alpha_3.$$

Now, for *every* angle after the flip there is at least one smaller angle before the flip:

$$\begin{aligned} \alpha_1 &> \underline{\alpha_1}, \\ \alpha_2 &> \underline{\alpha_2}, \\ \overline{\alpha_3} &> \alpha_3, \\ \overline{\alpha_4} &> \alpha_4, \\ \underline{\alpha_1} + \alpha_4 &> \alpha_4, \\ \underline{\alpha_2} + \alpha_3 &> \alpha_3. \end{aligned}$$

It follows that the smallest angle strictly increases. □

Theorem 5.19 (Inscribed Angle Theorem) *Let* C *be a circle with center* c *and positive radius and* $p, q \in C$*. Then the angle* $\angle prq \bmod \pi = \frac{1}{2}\angle pcq$ *is the same, for all* $r \in C$*.*

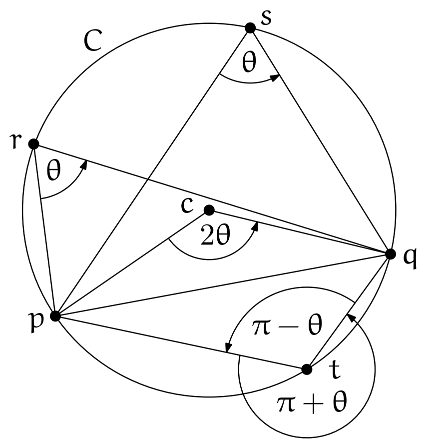
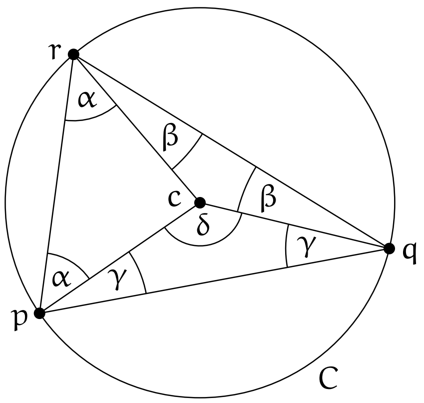


Figure 5.14: *The Inscribed Angle Theorem with* $\theta := \angle prq$.

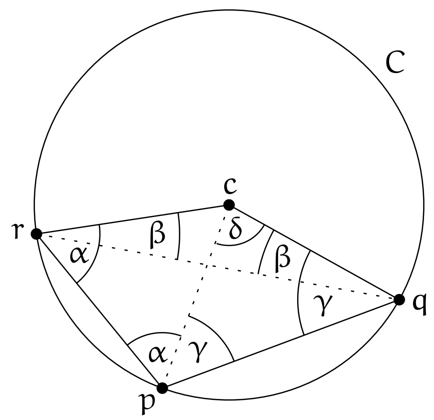
Proof. Without loss of generality we may assume that c is located to the left of or on the oriented line pq.

Consider first the case that the triangle $\Delta = pqr$ contains c. Then Δ can be partitioned into three triangles: pcr, qcr, and cpq. All three triangles are isosceles, because two sides of each form the radius of C. Denote $\alpha = \angle prc$, $\beta = \angle crq$, $\gamma = \angle cpq$, and $\delta = \angle pcq$ (see the figure shown to the right). The angles we are interested in are $\theta = \angle prq = \alpha + \beta$ and δ, for which we have to show that $\delta = 2\theta$.



Indeed, the angle sum in Δ is $\pi = 2(\alpha + \beta + \gamma)$ and the angle sum in the triangle cpq is $\pi = \delta + 2\gamma$. Combining both yields $\delta = 2(\alpha + \beta) = 2\theta$.

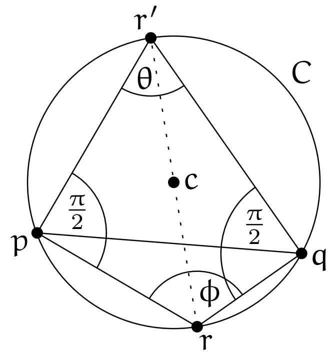
Next suppose that pqcr are in convex position and r is to the left of or on the oriented line pq. Without loss of generality let r be to the left of or on the oriented line qc. (The case that r lies to the right of or on the oriented line pc is symmetric.) Define α, β, γ, δ as above and observe that $\theta = \alpha - \beta$. Again have to show that $\delta = 2\theta$.



The angle sum in the triangle cpq is $\pi = \delta + 2\gamma$ and the angle sum in the triangle rpq is $\pi = (\alpha - \beta) + \alpha + \gamma + (\gamma - \beta) = 2(\alpha + \gamma - \beta)$. Combining both yields $\delta = \pi - 2\gamma = 2(\alpha - \beta) = 2\theta$.

It remains to consider the case that r is to the right of the oriented line pq.

Consider the point r' that is antipodal to r on C, and the quadrilateral $Q = prqr'$. We are interested in the angle ϕ of Q at r. By Thales' Theorem the inner angles of Q at p and q are both $\pi/2$. Hence the angle sum of Q is $2\pi = \theta + \phi + 2\pi/2$ and so $\phi = \pi - \theta$.



□

What happens in the case where the Delaunay triangulation is not unique? The following still holds.

Theorem 5.20 *Let $P \subseteq \mathbb{R}^2$ be a finite set of points, not all on a line. Every Delaunay triangulation $\mathcal{D}$ of P maximizes the smallest angle among all triangulations $\mathcal{T}$ of P.*

Proof. Let $\mathcal{D}$ be some Delaunay triangulation of P. We infinitesimally perturb the points in P such that no four are on a common circle anymore. Then the Delaunay triangulation becomes unique (Corollary 5.17). Starting from $\mathcal{D}$, we keep applying Lawson flips until we reach the unique Delaunay triangulation $\mathcal{D}^*$ of the perturbed point set. Now we examine this sequence of flips on the original *unperturbed* point set. All these flips must involve four cocircular points (only in the cocircular case, an infinitesimal perturbation can change "good" edges into "bad" edges that still need to be flipped). But as Figure 5.13 (a) easily implies, such a "degenerate" flip does not change the smallest of the six involved angles. It follows that $\mathcal{D}$ and $\mathcal{D}^*$ have the same smallest angle, and since $\mathcal{D}^*$ maximizes the smallest angle among all triangulations $\mathcal{T}$ (Theorem 5.18), so does $\mathcal{D}$. □

5.7 Constrained Triangulations

Sometimes one would like to have a Delaunay triangulation, but certain edges are already prescribed, for example, a Delaunay triangulation of a simple polygon. Of course, one cannot expect to be able to get a proper Delaunay triangulation where all triangles satisfy the empty circle property. But it is possible to obtain some triangulation that comes as close as possible to a proper Delaunay triangulation, given that we are forced to include the edges in E. Such a triangulation is called a *constrained Delaunay triangulation*, a formal definition of which follows.

Let $P \subseteq \mathbb{R}^2$ be a finite point set and $G = (P, E)$ a geometric graph with vertex set P (we consider the edges $e \in E$ as line segments). A triangulation $\mathcal{T}$ of P *respects* G if it contains all segments $e \in E$. A triangulation $\mathcal{T}$ of P that respects G is said to be a *constrained Delaunay triangulation* of P with respect to G if the following holds for every triangle Δ of $\mathcal{T}$:

The circumcircle of Δ contains only points $q \in P$ in its interior that are not visible from the interior of Δ. A point $q \in P$ is *visible* from the interior of Δ if there exists a point p in the interior of Δ such that the line segment $\overline{pq}$ does not intersect any segment $e \in E$. We can thus imagine the line segments of E as "blocking the view".

For illustration, consider the simple polygon and its constrained Delaunay triangulation shown in Figure 5.15. The circumcircle of the shaded triangle Δ contains a whole other triangle in its interior. But these points cannot be seen from Δ°, because all possible connecting line segments intersect the blocking polygon edge e of Δ.

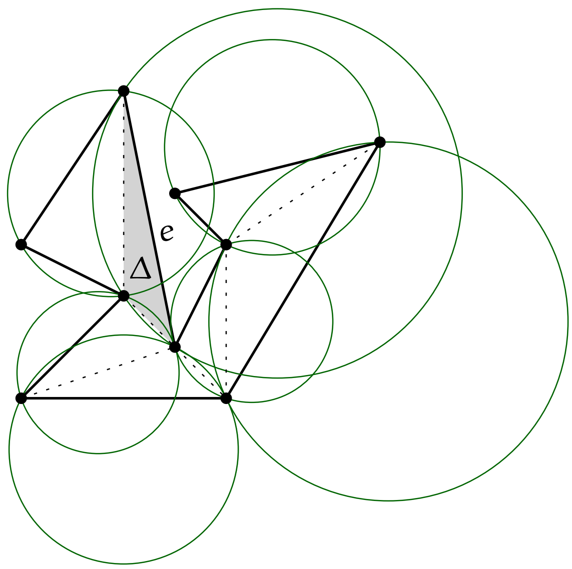


Figure 5.15: *Constrained Delaunay triangulation of a simple polygon.*

Theorem 5.21 *For every finite point set* P *and every plane graph* $G = (P, E)$*, there exists a constrained Delaunay triangulation of* P *with respect to* G.

Exercise 5.22 *Prove Theorem 5.21. Also describe a polynomial algorithm to construct such a triangulation.*

Questions

18. *What is a triangulation?* Provide the definition and prove a basic property: every triangulation with the same set of vertices and the same outer face has the same number of triangles.

19. *What is a triangulation of a point set?* Give a precise definition.

20. *Does every point set (not all points on a common line) have a triangulation?* You may, for example, argue with the scan triangulation.

21. *What is a Delaunay triangulation of a set of points*? Give a precise definition.

22. *What is the Delaunay graph of a point set?* Give a precise definition and a characterization.

23. *How can you prove that every set of points (not all on a common line) has a Delaunay triangulation?* You can for example sketch the Lawson flip algorithm and the Lifting Map, and use these to show the existence.

24. *When is the Delaunay triangulation of a point set unique?* Show that general position is a sufficient condition. Is it also necessary?

25. *What can you say about the "quality" of a Delaunay triangulation?* Prove that every Delaunay triangulation maximizes the smallest interior angle in the triangulation, among the set of all triangulations of the same point set.