

to its three neighboring triangles. Compare both approaches. Discuss different scenarios where you would prefer one over the other. In particular, analyze the space requirements of both.

Connectivity serves as an important indicator for properties of planar graphs. Another example is the following famous theorem of Tutte that provides a sufficient condition for Hamiltonicity. Its proof is beyond the scope of our lecture.

**Theorem 2.30 (Tutte [24])** *Every 4-connected planar graph is Hamiltonian.*

Moreover, for a given 4-connected planar graph a Hamiltonian cycle can also be computed in linear time [7].

## 2.5 Compact straight-line drawings

As a next step we consider plane embeddings in the geometric setting, where every edge is drawn as a straight-line segment. A classical theorem of Wagner and Fáry states that this is not a restriction in terms of plane embeddability.

**Theorem 2.31 (Fáry [9], Wagner [25])** *Every planar graph has a plane straight-line embedding (that is, it is isomorphic to a plane straight-line graph).*

This statement is quite surprising, considering how much more freedom arbitrarily complex Jordan arcs allow compared to line segments, which are completely determined by their endpoints. In order to further increase the level of appreciation, let us note that a similar “straightening” is not possible, when fixing the point set on which the vertices are to be embedded: For a given planar graph  $G = (V, E)$  on  $n$  vertices and a given set  $P \subset \mathbb{R}^2$  of  $n$  points, one can always find a plane embedding of  $G$  that maps  $V$  to  $P$  [20]. However, this is not possible in general with a plane *straight-line* embedding. For instance,  $K_4$  does not admit a plane straight-line embedding onto a set of points that form a convex quadrilateral, such as a rectangle. In fact, it is NP-hard to decide whether a given planar graph admits a plane straight-line embedding onto a given point set [5].

Although Fáry-Wagner’s theorem has a nice inductive proof, we will not discuss it here. Instead we will prove a stronger statement that implies Theorem 2.31.

A very nice property of straight-line embeddings is that they are easy to represent: We need to store points/coordinates for the vertices only. From an algorithmic and complexity point of view the space needed by such a representation is important, because it appears in the input and output size of algorithms that work on embedded graphs. While the Fáry-Wagner Theorem guarantees the existence of a plane straight-line embedding for every planar graph, it does not provide bounds on the size of the coordinates used in the representation. But the following strengthening provides such bounds, by describing an algorithm that embeds (without crossings) a given planar graph on a linear size integer grid.

**Theorem 2.32 (de Fraysseix, Pach, Pollack [11])** *Every planar graph on  $n \geq 3$  vertices has a plane straight-line drawing on the  $(2n - 3) \times (n - 1)$  integer grid.*

**Canonical orderings.** The key concept behind the algorithm is the notion of a canonical ordering, which is a vertex order that allows to construct a plane drawing in a natural (hence canonical) way. Reading it backwards one may think of a shelling or peeling order that destructs the graph vertex by vertex from the outside. A canonical ordering also provides a succinct representation for the combinatorial embedding.

**Definition 2.33** *A plane graph is internally triangulated if it is biconnected and every bounded face is a (topological) triangle. Let  $G$  be an internally triangulated plane graph and  $C_o(G)$  its outer cycle. A permutation  $\pi = (v_1, v_2, \dots, v_n)$  of  $V(G)$  is a canonical ordering for  $G$ , if*

- (1)  $G_k$  is internally triangulated, for all  $k \in \{3, \dots, n\}$ ;
- (2)  $v_1 v_2$  is on the outer cycle  $C_o(G_k)$  of  $G_k$ , for all  $k \in \{3, \dots, n\}$ ;
- (3)  $v_{k+1}$  is located in the outer face of  $G_k$  and its neighbors appear consecutively along  $C_o(G_k)$ , for all  $k \in \{3, \dots, n - 1\}$ ;

where  $G_k$  is the subgraph of  $G$  induced by  $v_1, \dots, v_k$ .

Figure 2.18 shows an example. Note that there are permutations that do not correspond to a canonical order: for instance, when choosing the vertex 4 as the eighth vertex instead of 8 in Figure 2.18b the graph  $G_8$  is not biconnected (1 is a cut-vertex).

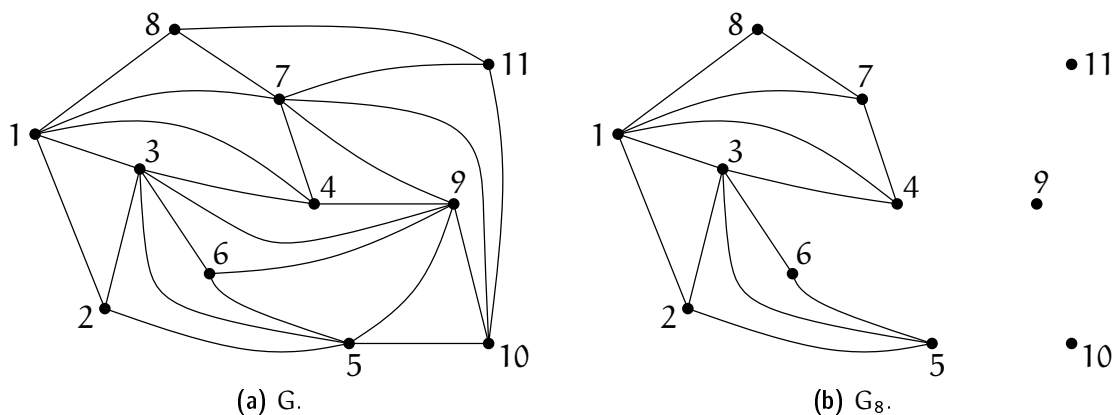
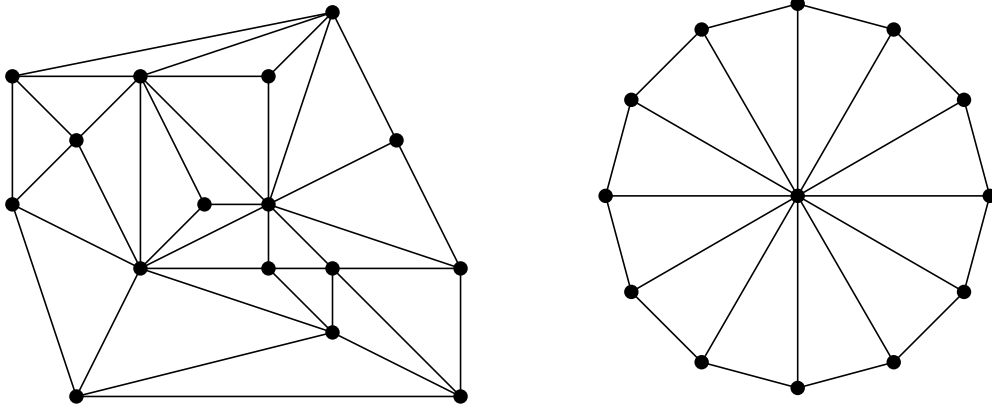


Figure 2.18: An internally triangulated plane graph and a canonical order for it.

**Exercise 2.34** (a) *Compute a canonical ordering for the following internally triangulated plane graphs:*



(b) Give a family of internally triangulated plane graphs  $G_n$  on  $n = 2k$  vertices with at least  $(n/2)!$  canonical orderings.

**Exercise 2.35** (a) Describe a plane graph  $G$  with  $n$  vertices that can be embedded (while preserving the outer face) on a grid of size  $(2n/3 - 1) \times (2n/3 - 1)$  but not on a smaller grid.

(b) Can you draw  $G$  on a smaller grid if you are allowed to change the embedding?

**Theorem 2.36** For every internally triangulated plane graph  $G$  and every edge  $\{v_1, v_2\}$  on its outer face, there exists a canonical ordering for  $G$  that starts with  $v_1, v_2$ . Moreover, such an ordering can be computed in linear time.

**Proof.** Induction on  $n$ , the number of vertices. For a triangle, any order suffices and so the statement holds. Hence consider an internally triangulated plane graph  $G = (V, E)$  on  $n \geq 4$  vertices. We claim that it is enough to select a vertex  $v_n \notin \{v_1, v_2\}$  on  $C_o(G)$  that is not incident to a chord of  $C_o(G)$ .

First observe that  $G$  is plane and  $v_n \in C_o(G)$  and so all neighbors of  $v_n$  in  $G$  must appear on the outer face of  $G_{n-1} = G \setminus \{v_n\}$ . Consider the circular sequence of neighbors around  $v_n$  in  $G$  and break it into a linear sequence  $u_1, \dots, u_m$ , for some  $m \geq 2$ , that starts and ends with the neighbors of  $v_n$  in  $C_o(G)$ . As  $G$  is internally triangulated, each of the bounded faces spanned by  $v_n, u_i, u_{i+1}$ , for  $i \in \{1, \dots, m-1\}$ , is a triangle and hence  $\{u_i, u_{i+1}\} \in E$ . This implies (3) for  $k = n$ . Properties (1) and (2) hold trivially (by assumption) in that case. In order to complete the ordering inductively we need to show that  $G_{n-1}$  is also internally triangulated.

The way  $G_{n-1}$  is obtained from  $G$ , every bounded face  $f$  of  $G_{n-1}$  also appears as a bounded face of  $G$ . As  $G$  is internally triangulated,  $f$  is a triangle. It remains to show that  $G_{n-1}$  is biconnected. The outer cycle  $C_o(G_{n-1})$  of  $G_{n-1}$  is obtained from  $C_o(G)$  by removing  $v_n$  and replacing it with the (possibly empty) sequence  $u_2, \dots, u_{m-1}$ . As  $v_n$  is not incident to a chord of  $C_o(G)$  (and so neither of  $u_2, \dots, u_{m-1}$  appeared along

$C_o(G)$  already), the resulting sequence forms a cycle, indeed. Add a new vertex  $v$  in the outer face of  $G_{n-1}$  and connect  $v$  to every vertex of  $C_o(G_{n-1})$  to obtain a maximal planar graph  $H \supset G_{n-1}$ . By Theorem 2.26  $H$  is 3-connected and so  $G_{n-1}$  is biconnected, as desired. This also completes the proof of the initial claim.

It remains to show that we can always find such a vertex  $v_n \notin \{v_1, v_2\}$  on  $C_o(G)$  that is not incident to a chord of  $C_o(G)$ . If  $C_o(G)$  does not have any chord, this is obvious, because every cycle has at least three vertices, one of which is neither  $v_1$  nor  $v_2$ . So suppose that  $C_o(G)$  has a chord  $c$ . The endpoints of  $c$  split  $C_o(G)$  into two paths, one of which does not have  $v_1$  nor  $v_2$  as an internal vertex. Among all possible chords of  $C_o(G)$  select  $c$  such that this path has minimal length. (It has always at least two edges, because there is always at least one vertex “behind” a chord.) Then by definition of  $c$  this path is an induced path in  $G$  and none of its (at least one) interior vertices is incident to a chord of  $C_o(G)$ , because such a chord would cross  $c$ . So we can select  $v_n$  from these vertices. By the way the path is selected with respect to  $c$ , this procedure does not select  $v_1$  nor  $v_2$ .

Regarding the runtime bound, we maintain the following information for each vertex  $v$ : whether it has been chosen already, whether it is on the outer face of the current graph, and the number of incident chords with respect to the current outer cycle. Given a combinatorial embedding of  $G$ , it is straightforward to initialize this information in linear time. (Every edge is considered at most twice, once for each endpoint on the outer face.)

When removing a vertex, there are two cases: Either  $v_n$  has two neighbors  $u_1$  and  $u_2$  only (Figure 2.19a), in which case the edge  $u_1u_2$  ceases to be a chord. Thus, the chord count for  $u_1$  and  $u_2$  has to be decremented by one. Otherwise, there are  $m \geq 3$  neighbors  $u_1, \dots, u_m$  (Figure 2.19b) and (1) all vertices  $u_2, \dots, u_{m-1}$  are new on the outer cycle, and (2) every edge incident to  $u_i$ , for  $i \in \{2, \dots, m-1\}$ , and some other vertex on the outer cycle other than  $u_{i-1}$  or  $u_{i+1}$  is a new chord (and the corresponding counters at the endpoints have to be incremented by one).

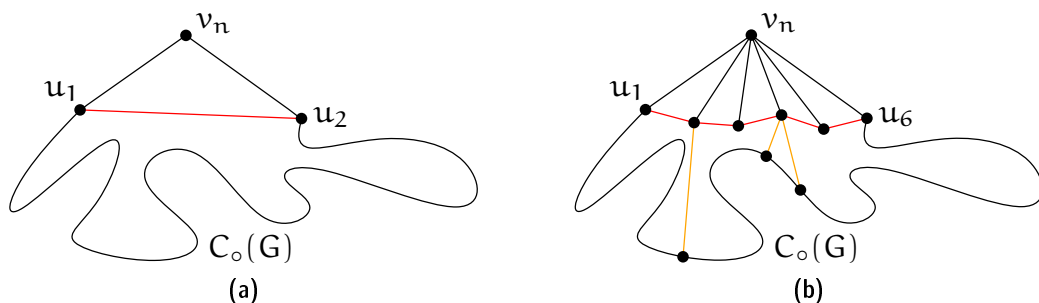


Figure 2.19: *Processing a vertex when computing a canonical ordering.*

During the course of the algorithm every vertex appears once as a new vertex on the outer face. At this point all incident edges are examined. Overall, every edge is inspected at most twice—once for each endpoint—which takes linear time by Corollary 2.5.  $\square$

Using one of the linear time planarity testing algorithms, we can obtain a combinatorial embedding for a given maximal planar graph  $G$ . As every maximal plane graph is internally triangulated, we can then use Theorem 2.36 to provide us with a canonical ordering for  $G$ , in overall linear time.

**Corollary 2.37** *Every maximal planar graph admits a canonical ordering. Moreover, such an ordering can be computed in linear time.*  $\square$

As simple as they may appear, canonical orderings are a powerful and versatile tool to work with plane graphs. As an example, consider the following partitioning theorem.

**Theorem 2.38 (Schnyder [22])** *For every maximal planar graph  $G$  on at least three vertices and every fixed face  $f$  of  $G$ , the multigraph obtained from  $G$  by doubling the (three) edges of  $f$  can be partitioned into three spanning trees.*

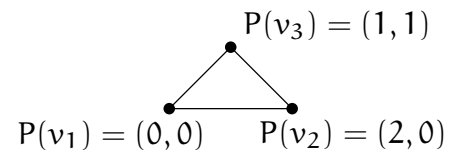
**Exercise 2.39** *Prove Theorem 2.38. Hint: Take a canonical ordering and build one tree by taking for every vertex  $v_k$  the edge to its first neighbor on the outer cycle  $C_o(G_{k-1})$ .*

Of a similar flavor is the following open problem, for which only partial answers for specific types of point sets are known [3, 1].

**Problem 2.40 (In memoriam Ferran Hurtado (1951–2014))**

Can every complete geometric graph on  $n = 2k$  vertices (the complete straight line graph on a set of  $n$  points in general position) be partitioned into  $k$  plane spanning trees?

**The shift-algorithm.** Let  $(v_1, \dots, v_n)$  be a canonical ordering. The general plan is to construct an embedding by inserting vertices in this order, starting from the triangle  $P(v_1) = (0, 0)$ ,  $P(v_3) = (1, 1)$ ,  $P(v_2) = (2, 0)$ . At each step, some vertices will be shifted to the right, making room for the edges to the freshly inserted vertex. For each vertex  $v_i$  already embedded, maintain a set  $L(v_i)$  of vertices that move rigidly together with  $v_i$ . Initially,  $L(v_i) = \{v_i\}$ , for  $1 \leq i \leq 3$ .



Ensure that the following invariants hold after Step  $k$  (that is, after  $v_k$  has been inserted):

- (i)  $P(v_1) = (0, 0)$ ,  $P(v_2) = (2k - 4, 0)$ ;
- (ii) The  $x$ -coordinates of the points on  $C_o(G_k) = (w_1, \dots, w_t)$  are strictly increasing (in this order)<sup>4</sup>;

<sup>4</sup>The notation is a bit sloppy here because both  $t$  and the  $w_i$  in general depend on  $k$ . So in principle we should write  $w_i^k$  instead of  $w_i$ . But as the  $k$  would just make a constant appearance throughout, we omit it to avoid index clutter.

(iii) each edge of  $C_o(G_k)$  is drawn as a straight-line segment with slope  $\pm 1$ .

Clearly these invariants hold for  $G_3$ , embedded as described above. Invariant (i) implies that after Step  $n$  we have  $P(v_2) = (2n - 4, 0)$ , while (iii) implies that the Manhattan distance<sup>5</sup> between any two points on  $C_o(G_k)$  is even.

Idea: put  $v_{k+1}$  at  $\mu(w_p, w_q)$ , where  $w_p, \dots, w_q$  are its neighbors on  $C_o(G_k)$  (recall that they appear consecutively along  $C_o(G_k)$  by definition of a canonical ordering), where

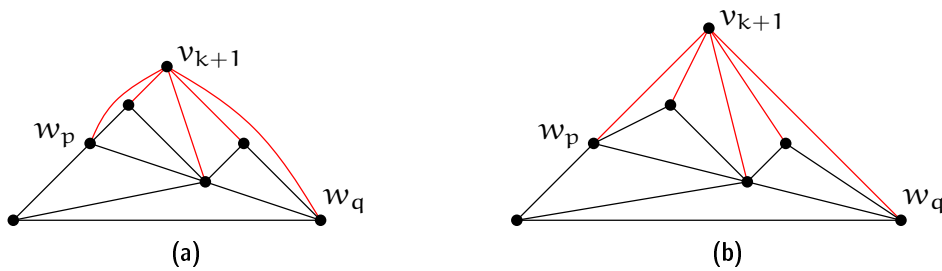
$$\mu((x_p, y_p), (x_q, y_q)) = \frac{1}{2}(x_p - y_p + x_q + y_q, -x_p + y_p + x_q + y_q)$$

is the point of intersection between the line  $\ell_1 : y = x - x_p + y_p$  of slope 1 through  $w_p = (x_p, y_p)$  and the line  $\ell_2 : y = x_q - x + y_q$  of slope  $-1$  through  $w_q = (x_q, y_q)$ .

**Proposition 2.41** *If the Manhattan distance between  $w_p$  and  $w_q$  is even, then  $\mu(w_p, w_q)$  is on the integer grid.*

**Proof.** By Invariant (ii) we know that  $x_p < x_q$ . Suppose without loss of generality that  $y_p \leq y_q$ . The Manhattan distance  $d$  of  $w_p$  and  $w_q$  is  $x_q - x_p + y_q - y_p$ , which by assumption is an even number. Adding the even number  $2x_p$  to  $d$  yields the even number  $x_q + x_p + y_q - y_p$ , half of which is the  $x$ -coordinate of  $\mu((x_p, y_p), (x_q, y_q))$ . Adding the even number  $2y_p$  to  $d$  yields the even number  $x_q - x_p + y_q + y_p$ , half of which is the  $y$ -coordinate of  $\mu((x_p, y_p), (x_q, y_q))$ .  $\square$

After Step  $n$  we have  $P(v_n) = (n - 2, n - 2)$ , because  $v_n$  is a neighbor of both  $v_1$  and  $v_2$ . However,  $P(v_{k+1})$  may not “see” all of  $w_p, \dots, w_q$ , in case that the slope of  $w_p w_{p+1}$  is 1 and/or the slope of  $w_{q-1} w_q$  is  $-1$  (Figure 2.20).



**Figure 2.20:** (a) The new vertex  $v_{k+1}$  is adjacent to all of  $w_p, \dots, w_q$ . If we place  $v_{k+1}$  at  $\mu(w_p, w_q)$ , then some edges may overlap, in case that  $w_{p+1}$  lies on the line of slope 1 through  $w_p$  or  $w_{q-1}$  lies on the line of slope  $-1$  through  $w_q$ ; (b) shifting  $w_{p+1}, \dots, w_{q-1}$  by one and  $w_q, \dots, w_t$  by two units to the right solves the problem.

In order to resolve these problems we shift some points around so that after the shift  $w_{p+1}$  does not lie on the line of slope 1 through  $w_p$  and  $w_{q-1}$  does not lie on the line of slope  $-1$  through  $w_q$ . The process of inserting  $v_{k+1}$  then looks as follows.

<sup>5</sup>The Manhattan distance of two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $|x_2 - x_1| + |y_2 - y_1|$ .

1. Shift  $\bigcup_{i=p+1}^{q-1} L(w_i)$  to the right by one unit.
2. Shift  $\bigcup_{i=q}^t L(w_i)$  to the right by two units.
3.  $P(v_{k+1}) := \mu(w_p, w_q)$ .
4.  $L(v_{k+1}) := \{v_{k+1}\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$ .

Observe that the Manhattan distance between  $w_p$  and  $w_q$  remains even, because the shift increases their  $x$ -difference by two and leaves the  $y$ -coordinates unchanged. Therefore by Proposition 2.41 the vertex  $v_{k+1}$  is embedded on the integer grid.

The slopes of the edges  $w_p w_{p+1}$  and  $w_{q-1} w_q$  (might be just a single edge, in case that  $p+1 = q$ ) become  $< 1$  in absolute value, whereas the slopes of all other edges along the outer cycle remain  $\pm 1$ . As all edges from  $v_{k+1}$  to  $w_{p+1}, \dots, w_{q-1}$  have slope  $> 1$  in absolute value, and the edges  $v_{k+1} w_p$  and  $v_{k+1} w_q$  have slope  $\pm 1$ , each edge  $v_{k+1} w_i$ , for  $i \in \{p, \dots, q\}$  intersects the outer cycle in exactly one point, which is  $w_i$ . In other words, adding all edges from  $v_{k+1}$  to its neighbors in  $G_k$  as straight-line segments results in a plane drawing.

Next we argue that the invariants (i)–(iii) are maintained. For (i) note that we start shifting with  $w_{p+1}$  only so that even in case that  $v_1$  is a neighbor of  $v_{k+1}$ , it is never shifted. On the other hand,  $v_2$  is always shifted by two, because we shift every vertex starting from (and including)  $w_q$ . Clearly both the shifts and the insertion of  $v_{k+1}$  maintain the strict order along the outer cycle, and so (ii) continues to hold. Finally, regarding (iii) note that the edges  $w_p w_{p+1}$  and  $w_{q-1} w_q$  (possibly this is just a single edge) are the only edges on the outer cycle whose slope is changed by the shift. But these edges do not appear on  $C_o(G_{k+1})$  anymore. The two edges  $v_{k+1} w_p$  and  $v_{k+1} w_q$  incident to the new vertex  $v_{k+1}$  that appear on  $C_o(G_{k+1})$  have slope 1 and  $-1$ , respectively. So all of (i)–(iii) are invariants of the algorithm, indeed.

So far we have argued about the shift with respect to vertices on the outer cycle of  $G_k$  only. To complete the proof of Theorem 2.32 it remains to show that the drawing remains plane under shifts also in its interior part.

**Lemma 2.42** *Let  $G_k$ ,  $3 \leq k \leq n$ , be straight-line grid embedded as described,  $C_o(G_k) = (w_1, \dots, w_t)$ , and let  $\delta_1 \leq \dots \leq \delta_t$  be non-negative integers. If for each  $i$ , we shift  $L(w_i)$  by  $\delta_i$  to the right, then the resulting straight-line drawing is plane.*

**Proof.** Induction on  $k$ . For  $G_3$  this is obvious. Let  $v_k = w_\ell$ , for some  $1 < \ell < t$ . Construct a delta sequence  $\Delta$  for  $G_{k-1}$  as follows. If  $v_k$  has only two neighbors in  $G_k$ , then  $C_o(G_{k-1}) = (w_1, \dots, w_{\ell-1}, w_{\ell+1}, \dots, w_t)$  and we set  $\Delta = \delta_1, \dots, \delta_{\ell-1}, \delta_{\ell+1}, \dots, \delta_t$ . Otherwise,  $C_o(G_{k-1}) = (w_1, \dots, w_{\ell-1} = u_1, \dots, u_m = w_{\ell+1}, \dots, w_t)$ , where  $u_1, \dots, u_m$  are the  $m \geq 3$  neighbors of  $v_k$  in  $G_k$ . In this case we set

$$\Delta = \delta_1, \dots, \delta_{\ell-1}, \underbrace{\delta_\ell, \dots, \delta_\ell}_{m-2 \text{ times}}, \delta_{\ell+1}, \dots, \delta_t.$$

Clearly,  $\Delta$  is monotonely increasing and by the inductive assumption a correspondingly shifted drawing of  $G_{k-1}$  is plane. When adding  $v_k$  and its incident edges back, the drawing remains plane: All vertices  $u_2, \dots, u_{m-1}$  (possibly none) move rigidly with (by exactly the same amount as)  $v_k$  by construction. Stretching the edges of the chain  $w_{\ell-1}, w_\ell, w_{\ell+1}$  by moving  $w_{\ell-1}$  to the left and/or  $w_{\ell+1}$  to the right cannot create any crossings.  $\square$

**Linear time.** (*This part was not covered in the lecture.*) The challenge in implementing the shift algorithm efficiently lies in the eponymous shift operations, which modify the  $x$ -coordinates of potentially many vertices. In fact, it is not hard to see that a naive implementation—which keeps track of all coordinates explicitly—may use quadratic time. De Fraysseix et al. described an implementation of the shift algorithm that uses  $O(n \log n)$  time. Then Chrobak and Payne [8] observed how to improve the runtime to linear, using the following ideas.

Recall that  $P(v_{k+1}) = (x_{k+1}, y_{k+1})$ , where

$$\begin{aligned} x_{k+1} &= \frac{1}{2}(x_p - y_p + x_q + y_q) \text{ and} \\ y_{k+1} &= \frac{1}{2}(-x_p + y_p + x_q + y_q) = \frac{1}{2}((x_q - x_p) + y_p + y_q). \end{aligned} \tag{2.43}$$

Thus,

$$x_{k+1} - x_p = \frac{1}{2}((x_q - x_p) + y_q - y_p). \tag{2.44}$$

$\Rightarrow$  We need the  $y$ -coordinates of  $w_p$  and  $w_q$  together with the relative  $x$ -position (offset) of  $w_p$  and  $w_q$  only in to determine the  $y$ -coordinate of  $v_{k+1}$  and its offset to  $w_p$ .

Maintain the outer cycle as a rooted binary tree  $T$ , with root  $v_1$ . For each node  $v$  of  $T$ , the *left child* is the first vertex covered by insertion of  $v$  (if any), that is,  $w_{p+1}$  in the terminology from above (if  $p + 1 \neq q$ ), whereas the *right child* of  $v$  is the next node along the outer cycle (if any; either along the current outer cycle or along the one at the point where both points were covered together). See Figure 2.21 for an example.

At each node  $v$  of  $T$  we also store its  $x$ -offset  $dx(v)$  with respect to the parent node. For the root  $v_1$  of the tree set  $dx(v_1) = 0$ . In this way, a whole subtree (and, thus, a whole set  $L(\cdot)$ ) can be shifted by changing a single offset entry at its root.

Initially,  $dx(v_1) = 0$ ,  $dx(v_2) = dx(v_3) = 1$ ,  $y(v_1) = y(v_2) = 0$ ,  $y(v_3) = 1$ ,  $left(v_1) = left(v_2) = left(v_3) = 0$ ,  $right(v_1) = v_3$ ,  $right(v_2) = 0$ , and  $right(v_3) = v_2$ .

Inserting a vertex  $v_{k+1}$  works as follows. As before, let  $w_1, \dots, w_t$  denote the vertices on the outer cycle  $C_o(G_k)$  and  $w_p, \dots, w_q$  be the neighbors of  $v_{k+1}$ .

1. Increment  $dx(w_{p+1})$  and  $dx(w_q)$  by one. (*This implements the shift.*)
2. Compute  $\Delta_{pq} = \sum_{i=p+1}^q dx(w_i)$ . (*This is the total offset between  $w_p$  and  $w_q$ .*)



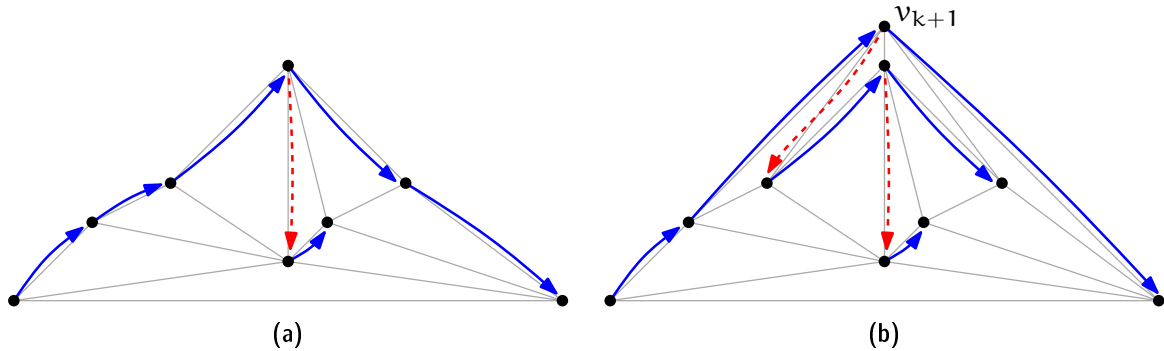


Figure 2.21: *Maintaining the binary tree representation when inserting a new vertex  $v_{k+1}$ . Red (dashed) arrows point to left children, blue (solid) arrows point to right children.*

3. Set  $dx(v_k) \leftarrow \frac{1}{2}(\Delta_{pq} + y(w_q) - y(w_p))$  and  $y(v_k) \leftarrow \frac{1}{2}(\Delta_{pq} + y(w_q) + y(w_p))$ .  
(This is exactly what we derived in (2.43) and (2.44).)
4. Set  $right(w_p) \leftarrow v_k$  and  $right(v_k) \leftarrow w_q$ . (Update the current outer cycle.)
5. If  $p+1 = q$ , then set  $left(v_k) \leftarrow 0$ ; else set  $left(v_k) \leftarrow w_{p+1}$  and  $right(w_{q-1}) \leftarrow 0$ .  
(Update the part that is covered by insertion of  $v_{k+1}$ .)
6. Set  $dx(w_q) \leftarrow \Delta_{pq} - dx(v_k)$  and—unless  $p+1 = q$ —set  $dx(w_{p+1}) \leftarrow dx(w_{p+1}) - dx(v_k)$ . (Update the offsets according to the changes in the previous two steps.)

Observe that the only step that possibly cannot be executed in constant time is Step 2. But all vertices but the last vertex  $w_q$  for which we sum the offsets are covered by the insertion of  $v_{k+1}$ . As every vertex can be covered at most once, the overall complexity of this step during the algorithm is linear. Therefore, this first phase of the algorithm can be completed in linear time.

In a second phase, the final  $x$ -coordinates can be computed from the offsets by a single recursive pre-order traversal of the tree. The (pseudo-)code given below is to be called with the root vertex  $v_1$  and an offset of zero. Clearly this yields a linear time algorithm overall.

```

compute_coordinate(Vertex v, Offset d) {
  if (v == 0) return;
  x(v) = dx(v) + d;
  compute_coordinate(left(v), x(v));
  compute_coordinate(right(v), x(v));
}

```

**Remarks.** From a geometric complexity point of view, Theorem 2.32 provides very good news for planar graphs in a similar way that the Euler Formula does from a combinatorial

complexity point of view. Euler's Formula tells us that we can obtain a combinatorial representation (for instance, as a DCEL) of any plane graph using  $O(n)$  space, where  $n$  is the number of vertices.

Now the shift algorithm tells us that for any planar graph we can even find a geometric plane (straight-line) representation using  $O(n)$  space. In addition to the combinatorial information, we only have to store  $2n$  numbers from the range  $\{0, 1, \dots, 2n - 4\}$ .

When we make such claims regarding space complexity we implicitly assume the so-called *word RAM model*. In this model each address in memory contains a *word* of  $b$  bits, which means that it can be used to represent any integer from  $\{0, \dots, 2^b - 1\}$ . One also assumes that  $b$  is sufficiently large, for instance, in our case  $b \geq \log n$ .

There are also different models such as the *bit complexity model*, where one is charged for every bit used to store information. In our case that would already incur an additional factor of  $\log n$  for the combinatorial representation: for instance, for each halfedge we store its endpoint, which is an index from  $\{1, \dots, n\}$ .

## Questions

1. *What is an embedding? What is a planar/plane graph?* Give the definitions and explain the difference between planar and plane.
2. *How many edges can a planar graph have? What is the average vertex degree in a planar graph?* Explain Euler's formula and derive your answers from it.
3. *How can plane graphs be represented on a computer?* Explain the DCEL data structure and how to work with it.
4. *How can a given plane graph be (topologically) triangulated efficiently?* Explain what it is, including the difference between topological and geometric triangulation. Give a linear time algorithm, for instance, as in Theorem 2.25.
5. *What is a combinatorial embedding? When are two combinatorial embeddings equivalent? Which graphs have a unique combinatorial plane embedding?* Give the definitions, explain and prove Whitney's Theorem.
6. *What is a canonical ordering and which graphs admit such an ordering? For a given graph, how can one find a canonical ordering efficiently?* Give the definition. State and prove Theorem 2.36.
7. *Which graphs admit a plane embedding using straight line edges? Can one bound the size of the coordinates in such a representation?* State and prove Theorem 2.32.

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