ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Lecturer: B. Gärtner

Informatik für Mathematiker und Physiker HS12

Exercise Sheet 13

Submission deadline: 3.15pm - Tuesday 18th December, 2012

Course URL: http://www.ti.inf.ethz.ch/ew/Lehre/Info1_12/

Assignment 1 - (4 points)

Consider the probability space associated wit rolling a black and a white dice:

$$\Omega = \{(i, j) : 1 \leq i, j \leq 6\},$$

where i is the value of the black dice and j the value of the white dice. Moreover,

$$\Pr[(i, j)] = \frac{1}{36}, \quad 1 \leq i, j \leq 6.$$

Let $X_{\max}$ be the random variable corresponding to the maximum of the two dice, i.e. $X_{\max}(i, j) = \max\{i, j\}$. Similarly, $X_{\min}$ is the random variable for the miminum, $X_{\min}(i, j) = \min\{i, j)\}$. Compute the expectations $\mathbf{E}[X_{\max}]$ and $\mathbf{E}[X_{\min}]$.

Assignment 2 - (4 + 1 + 3 + 4 points)

Let Ω be the set of all sequences $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that $\{\pi_1, \pi_2, \ldots, \pi_n\} = \{1, 2, \ldots, n\}$. Such a sequence is also known as a *permutation* of $\{1, 2, \ldots, n\}$, and there are $n!$ of them. For example, if $n = 3$, there are the $3! = 6$ permutations:

$$123, 132, 231, 213, 312, 321.$$

For each $\pi \in \Omega$, we define $\Pr[\pi] = 1/n!$. The probability space $(\Omega, \Pr)$ corresponds to choosing a *random permutation* of $\{1, 2, \ldots, n\}$.

a) Let A_i be the event

$$A_i = \{\pi \in \Omega : \pi_i < \min\{\pi_1, \ldots, \pi_{i-1}\}\}, \quad i = 1, 2, \ldots, n,$$

where we define $\min(\emptyset) = \infty$. Prove that

$$\Pr[A_i] = \frac{1}{i}, \quad i = 1, 2, \ldots, n.$$

b) Let X_i be the *indicator* random variable of the event A_i, i.e.

$$X_i(\pi) = \begin{cases} 1, & \text{if } \pi \in A_i, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\mathbf{E}[X_i] = 1/i$ for $i = 1, 2, \ldots, n$.

c) Let $X(\pi)$ be the number of *prefix minima* in π, i.e. the number of elements that are smaller than all its predecessors. For example,

$$X(1, 2, \ldots, n) = 1, \quad X(n, n-1, \ldots, 1) = n.$$

Prove that $\mathbf{E}[X] = H_n$, the n-th Harmonic number.

d) From the course web page, download the file findmin.cpp, a program for finding the minimum element in a random permutation of n elements. Equip the program with a variable that computes the *average* number of prefix minima that are found while searching for the minimum, where the average is taken over 100 runs of the program. Make a table of these numbers for $n = 1, 10, 10^2, \ldots, 10^7$ and compare them with the expected value H_n of prefix minima (for the computation of H_n, refer to the program `harmonic.cpp`).

Challenge - (8 points)

In the lecture, we have analyzed the expected depth of the smallest key in a treap. What is the expected *total* depth, the expected sum of the depths of *all* n keys?