ETH Zürich
Institute of Theoretical Computer Science

Metric Embeddings
Spring 2010

Problem Set 1 (Deadline: March 8)

1. (a) Prove the *short diagonals lemma*: For every four points x_1, x_2, x_3, x_4 in Euclidean space of arbitrary dimension, we have

$$\|x_1 - x_3\|_2^2 + \|x_2 - x_4\|_2^2 \leq \|x_1 - x_2\|_2^2 + \|x_2 - x_3\|_2^2 + \|x_3 - x_4\|_2^2 + \|x_4 - x_1\|_2^2.$$

   (b) Find the minimum necessary distortion for embedding the 4-cycle[1] into ℓ_2 (i.e., into a Euclidean space of arbitrary dimension).

2. (a) Show that every embedding of the complete graph K_n considered as a metric space into the plane $\mathbb{R}^2$ with the usual Euclidean metric has distortion at least[2] $\Omega(\sqrt{n})$.

   (b) Give an embedding with distortion $O(\sqrt{n})$.

3. Show that every embedding of the cycle of length n into the line $\mathbb{R}^1$ with the usual metric has distortion at least $\Omega(n)$. (Hint: Consider three vertices on the cycle separated by distances $n/3$.)

4. Show that every tree can be embedded isometrically into ℓ_1 (it's enough to do the unweighted case).

Food for thought (will be solved in class next week)

1. Let G_n be the unweighted graph obtained from K_5 by replacing every edge by a path of length n. Show that every embedding of G_n into plane $\mathbb{R}^2$ with the usual Euclidean metric has distortion $\Omega(n)$.

2. (a) Let S^2 denote the 2-dimensional unit sphere in $\mathbb{R}^3$. Let $X \subset S^2$ be a set of n points. Show that (X, ℓ_2) can be embedded into the Euclidean plane with distortion $O(\sqrt{n})$.

   (b) Show that there exists an n-point set $Y \subset S^2$ such that every embedding of $(Y, \|\cdot\|_2)$ into $(\mathbb{R}^2, \|\cdot\|_2)$ has distortion $\Omega(\sqrt{n})$. (Hint: The Borsuk-Ulam theorem states that for every continuous map $h : S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $h(x) = h(-x)$.)

[1] Whenever we mention a graph as a metric space, we mean the shortest-path metric on the vertex set, and unless stated otherwise, we assume that all edges have unit length.

[2] A reminder of asymptotic notation: $f(n) = O(g(n))$ means that there are n_0 and C such that $f(n) \leq Cg(n)$ for all $n \geq n_0$; $f(n) = o(g(n))$ means that $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Omega(g(n))$ is the same as $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ means that both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.