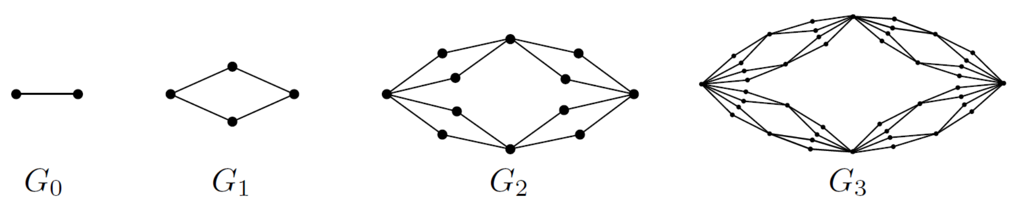
ETH Zürich
Institute of Theoretical Computer Science

Metric Embeddings
Spring 2010

Problem Set 3

(to be handed in April 12 in class)

1. The *k-th diamond graph* G_k is defined recursively as follows: G_0 is a single edge, and for $k \geq 1$, G_k is obtained from G_{k-1} by replacing each edge uv in G_{k-1} by a 4-cycle u, a, v, b, where a and b are new vertices:

   Show that for every k, G_k can be 2-embedded into ℓ_1. (Hint: Map each vertex of G_k to a point in $\{0,1\}^{2^k}$.)

2. (a) Prove that if G is a graph whose average vertex degree is d, then G contains a subgraph with minimum vertex degree at least $d/2$.

   (b) Show that every graph G has a bipartite subgraph H that contains at least half the edges of G.

   (c) Use (a) and (b) to prove that if $G = (V, E)$ is an n-vertex graph that does not contain any cycle of length ℓ or less, then $|E| = O(n^{1+1/\lfloor \ell/2 \rfloor})$.

3. (a) Let T be a tree on $n \geq 3$-vertices. Prove that there exist subtrees T_1 and T_2 of T that share a single vertex and no edges and together cover T, such that $\max\{|V(T_1)|, |V(T_2)|\} \leq 1 + \frac{2}{3}n$.

   (b) Using (a), prove that every n-point tree can be isometrically embedded into ℓ_∞^d with $d = O(\log n)$.