ETH Zürich **Metric Embeddings**
Institute of Theoretical Computer Science **Spring 2010**

Problem Set 4

(to be handed in May 3 in class)

1. A metric (V, ρ) is a *metric of negative type* if there exists a mapping $f : V \to \ell_2$ such that $\rho(u, v) = \|f(u) - f(v)\|_2^2$ for every $u, v \in V$.

   (a) Show that the ℓ_2-squared distances between the points of a set $X \subset \mathbb{R}^d$ form a metric of negative type if and only if no triple in X forms an obtuse angle.

   (b) Show that every finite path (as a graph metric) is a metric of negative type, by giving an explicit embedding.

2. Refer to the proof of Theorem 4.2.1 (Bourgain's theorem) in the lecture notes. Show that the same mapping $f : V \to \mathbb{R}^d$ as given in the proof also provides an embedding of V into ℓ_p with $O(\log n)$ distortion, for every fixed $p \in [1, \infty)$. Describe only the modifications of the proof—you need not repeat parts that remain unchanged.

   (Hint: Use Hölder's inequality.)

3. The "majority" boolean function $\mathrm{Maj} : \{0, 1\}^d \to \{0, 1\}$, for d odd, is given by

$$\mathrm{Maj}(u) = \begin{cases} 1, & \text{if } \sum_i u_i > d/2; \\ 0, & \text{otherwise.} \end{cases}$$

   Calculate the asymptotic influence $I_k(\mathrm{Maj})$ of a single coordinate k.

4. The boolean function $\mathrm{Tribes} = \mathrm{Tribes}_{d,t} : \{0, 1\}^d \to \{0, 1\}$ is given by

$$\mathrm{Tribes}(u) = (u_1 \wedge \cdots \wedge u_t) \vee (u_{t+1} \wedge \cdots \wedge u_{2t}) \vee (u_{2t+1} \wedge \cdots \wedge u_{3t}) \vee \cdots .$$

   (a) Calculate the value of t (as a function of d) that makes the function Tribes as balanced as possible.

   (b) For that value of t, what is the influence $I_k(\mathrm{Tribes})$ of a coordinate k?

5. Let $f, g : \{0, 1\}^d \to \{0, 1\}$ be boolean functions such that there exists a $w \in \{0, 1\}^d$ such that $g(u) = f(u + w)$ for all u. Prove that $\hat{g}(u) = (-1)^{<u,w>} \hat{f}(u)$.

6. Let P_n be the metric space $\{0, 1, \ldots, n\}$ with the metric inherited from $\mathbb{R}$. Prove the following Ramsey-type result: For every $D > 1$ and every $\varepsilon > 0$ there exists an $n = n(D, \varepsilon)$ such that whenever $f : P_n \to (Z, \sigma)$ is a D-embedding of P_n into some metric space, there are $a < b < c$, $b = \frac{a+c}{2}$, such that f restricted to the subspace $\{a, b, c\}$ of P_n is a $(1 + \varepsilon)$-embedding.

In other words, if a sufficiently long path (with the graph metric) is D-embedded, then it contains a scaled copy of a path of length 2 embeded with distortion close to 1.

(Hint: Consider the sequence $K_0, K_1, K_2, \ldots$, where

$$K_i = 2^{-i} \max_a \sigma(f(a), f(a + 2^i)).)$$