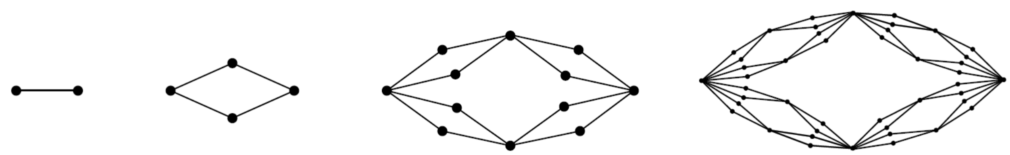
ETH Zürich | **Metric Embeddings**
Institute of Theoretical Computer Science | **Spring 2010**

Problem Set 6
(to be handed in May 31 in class)

1. Recall the diamond graph G_k from Problem Set 3:



G_0 G_1 G_2 G_3

Let (V, ρ) be the graph metric on G_k, and let $1 < p \leq 2$ be fixed.

(a) Show that there exist nonnegative coefficients a_{uv}, b_{uv} for $u, v \in V$ such that for every embedding $f : V \to \ell_p$,

$$\sum_{u,v \in V} a_{uv} \|f(u) - f(v)\|_p^2 \leq \sum_{u,v \in V} b_{uv} \|f(u) - f(v)\|_p^2,$$

$$\sum_{u,v \in V} a_{uv} \rho(u,v)^2 = D \sum_{u,v \in V} b_{uv} \rho(u,v)^2,$$

for $D = 1 + (p-1)k$.

(Hint: Repeatedly apply the *short-diagonals lemma for* ℓ_p, which states that for every four points x_1, x_2, x_3, $x_4 \in \ell_p$ we have

$$\|x_1 - x_3\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \leq \|x_1 - x_2\|_p^2 + \|x_2 - x_3\|_p^2 + \|x_3 - x_4\|_p^2 + \|x_4 - x_1\|_p^2.$$

This lemma is somewhat hard, and you do not have to prove it.)

(b) Conclude that every embedding of G_k into ℓ_p must have distortion at least $\sqrt{1 + (p-1)k}$.

2. Design a streaming algorithm that uses space $O(\log n)$ and solves the following problem. Let A be a set contining $n - 1$ distinct numbers from $\{1, \ldots, n\}$. The algorithm reads a stream containing A in an arbitrary order and outputs the missing number $x \in \{1, \ldots, n\} \setminus A$.

3. (a) Let $d \geq 1$. Give an isometric embedding of ℓ_1^d to $\ell_\infty^{2^d}$.

(b) Devise an algorithm that, given a set X of n points in $\mathbb{R}^d$, computes the diameter of X under the ℓ_1 norm in time $O(f(d) \cdot n)$, for some arbitrary function f.

4. Show that for every r and ℓ there exists an r-regular graph $G(r, \ell)$ of girth ℓ. (Hint: Use double induction; construct $G(r, \ell)$ from $G(r-1, \ell)$ and $G(r', \ell')$ for some $\ell' < \ell$ and some r'.)