Chapter 1
Theory of Convex Functions

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This chapter develops the basic theory of convex functions that we will need later. Much of the material is also covered in other courses, so we will refer to the literature for standard material and focus more on material that we feel is less standard (but important in our context).

1.1 Notation

For vectors in $\mathbb{R}^d$, we use bold font, and for their coordinates normal font, e.g. $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. $x_1, x_2, \ldots$ denotes a sequence of vectors. Vectors are considered as column vectors, unless they are explicitly transposed. $\|\mathbf{x}\|$ denotes the Euclidean norm ($\ell_2$-norm or 2-norm) of vector $\mathbf{x}$,

$$\|\mathbf{x}\| = \mathbf{x}^\top \mathbf{x} = \sum_{i=1}^{d} x_i^2.$$ 

We also use

$$\mathbb{N} = \{1, 2, \ldots\}, \quad \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}.$$ 

We are freely using basic notions and material such as open and closed sets, vector spaces, continuity, convergence, limits, triangle inequality, …

1.2 Convex sets

Definition 1.1. A set $C \subseteq \mathbb{R}^d$ is convex if for any two points $\mathbf{x}, \mathbf{y} \in C$, the connecting line segment is contained in $C$. In formulas, if for all $\lambda \in [0, 1]$, $\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in C$; see Figure 1.1.

![Figure 1.1: A convex set (left) and a non-convex set (right)](image-url)
Observation 1.2. Let $C_i, i \in I$ be convex sets, where $I$ is a (possibly infinite) index set. Then $C = \bigcap_{i \in I} C_i$ is a convex set.

1.3 Convex functions

We are considering real-valued functions $f : \text{dom}(f) \to \mathbb{R}$, where $\text{dom}(f) \subseteq \mathbb{R}^d$ denotes the domain of $f$. The graph of $f$ is the set $\{(x, f(x)) \in \mathbb{R}^{d+1} : x \in \text{dom}(f)\}$. The epigraph (Figure 1.2) is the set of points above the graph,

$$\text{epi}(f) := \{(x, \alpha) \in \mathbb{R}^{d+1} : x \in \text{dom}(f), \alpha \geq f(x)\}.$$

![Graph and epigraph](image)

Figure 1.2: Graph and epigraph of a non-convex function (left) and a convex function (right)

Definition 1.3 ([BV04, 3.1.1]). A function $f : \text{dom}(f) \to \mathbb{R}$ is convex if (i) $\text{dom}(f)$ is convex and (ii) for all $x, y \in \text{dom}(f)$ and all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1.1)$$

Geometrically, the condition means that the line segment connecting the two points $(x, f(x)), (y, f(y)) \in \mathbb{R}^{d+1}$ lies pointwise above the graph of $f$; see Figure 1.3. (Whenever we say “above”, we mean “above or on”.) An important special case arises when $f : \mathbb{R}^d \to \mathbb{R}$ is an affine function,
i.e. \( f(x) = c^T x + c_0 \) for some vector \( c \in \mathbb{R}^d \) and scalar \( c_0 \in \mathbb{R} \). In this case, (1.1) is always satisfied with equality, and line segments connecting points on the graph lie pointwise on the graph.

**Observation 1.4.** \( f \) is a convex function if and only if \( \text{epi}(f) \) is a convex set.

**Proof.** This is easy but let us still do it to illustrate the concepts. Let \( f \) be a convex function and consider two points \((x, \alpha), (y, \beta) \in \text{epi}(f), \lambda \in [0, 1]\). This means, \( f(x) \leq \alpha, f(y) \leq \beta \), hence by convexity of \( f \),

\[
f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda\alpha + (1-\lambda)\beta.
\]

Therefore, by definition of the epigraph,

\[
\lambda(x, \alpha) + (1-\lambda)(y, \beta) = (\lambda x + (1-\lambda)y, \lambda\alpha + (1-\lambda)\beta) \in \text{epi}(f),
\]

so \( \text{epi}(f) \) is a convex set. In the other direction, let \( \text{epi}(f) \) be a convex set and consider two points \( x, y \in \text{dom}(f), \lambda \in [0, 1]\). By convexity of \( \text{epi}(f) \), we have

\[
\text{epi}(f) \ni \lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)),
\]

and this is just a different way of writing (1.1). \( \square \)
Lemma 1.5 (Jensen’s inequality). Let $f$ be convex, $x_1, \ldots, x_m \in \text{dom}(f)$, $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$ such that $\sum_{i=1}^{m} \lambda_i = 1$. Then

$$f \left( \sum_{i=1}^{m} \lambda_i x_i \right) \leq \sum_{i=1}^{m} \lambda_i f(x_i).$$

For $m = 2$, this is (1.1). The proof of the general case is Exercise 1.

Lemma 1.6. Let $f$ be convex and suppose that $\text{dom}(f)$ is open. Then $f$ is continuous.

This is not entirely obvious (see Exercise 2), and it becomes false if we consider convex functions over general vector spaces. What saves us is that $\mathbb{R}^d$ has finite dimension.

As an example, let us consider $f(x_1, x_2) = x_1^2 + x_2^2$. The graph of $f$ is the unit paraboloid in $\mathbb{R}^3$ which looks convex. However, to verify (1.1) directly is somewhat cumbersome. Next, we develop better ways to do this.

### 1.3.1 First-order characterization of convexity

If $f$ is differentiable, convexity can be characterized as follows.

Lemma 1.7 ([BV04, 3.1.3]). Suppose that $\text{dom}(f)$ is open and that $f$ is differentiable; in particular, the gradient (vector of partial derivatives)

$$\nabla f(x) := \left( \frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_d} \right)$$

exists at every point $x \in \text{dom}(f)$. Then $f$ is convex if and only if $\text{dom}(f)$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) \quad \text{(1.2)}$$

holds for all $x, y \in \text{dom}(f)$.

Geometrically, this means that for all $x \in \text{dom}(f)$, the graph of $f$ lies above its tangent hyperplane at the point $(x, f(x))$; see Figure 1.4.

For $f(x_1, x_2) = x_1^2 + x_2^2$, we have $\nabla f(x) = (2x_1, 2x_2)$, hence (1.2) boils down to

$$y_1^2 + y_2^2 \geq x_1^2 + x_2^2 + 2x_1(y_1 - x_1) + 2x_2(y_2 - x_2),$$

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Figure 1.4: First-order characterization of convexity

which after some rearranging of terms is equivalent to

\[(y_1 - x_1)^2 + (y_2 - x_2)^2 \geq 0,\]

hence true. There are relevant convex functions that are not differentiable, see Figure 1.5 for an example. More generally, Exercise 8 asks you to prove that the \(\ell_1\)-norm (or 1-norm) \(f(x) = \|x\|_1\) is convex.

Figure 1.5: A non-differentiable convex function

1.3.2 Second-order characterization of convexity

If \(f\) is twice differentiable, convexity can be characterized as follows.
Lemma 1.8 ([BV04, 3.1.4]). Suppose that $\text{dom}(f)$ is open and that $f$ is twice differentiable; in particular, the Hessian (matrix of second partial derivatives)

$$\nabla^2 f(x) = \begin{pmatrix}
\frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_d} \\
\frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_d} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f(x)}{\partial x_d \partial x_1} & \frac{\partial^2 f(x)}{\partial x_d \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_d \partial x_d}
\end{pmatrix},$$

exists at every point $x \in \text{dom}(f)$ and is symmetric. Then $f$ is convex if and only if $\text{dom}(f)$ is convex, and for all $x \in \text{dom}(f)$, we have

$$\nabla^2 f(x) \succeq 0 \quad \text{(i.e. } \nabla^2 f(x) \text{ is positive semidefinite)}. \quad (1.3)$$

(A symmetric matrix $M$ is positive semidefinite if $x^T M x \geq 0$ for all $x$, and positive definite if $x^T M x > 0$ for all $x$.)

Geometrically, this means that the graph of $f$ has nonnegative curvature everywhere and hence “looks like a bowl”. For $f(x_1, x_2) = x_1^2 + x_2^2$, we have

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is a positive definite matrix. In higher dimensions, the same argument can be used to show that the squared distance $d_y(x) = \|x - y\|^2$ to a fixed point $y$ is a convex function; see Exercise 3. The non-squared Euclidean distance $\|x - y\|$ is also convex in $x$, as a consequence of Lemma 1.9(ii) below and the fact that every seminorm (in particular the Euclidean norm $\|x\|$) is convex (Exercise 9). The squared Euclidean distance has the advantage that it is differentiable, while the Euclidean distance itself (whose graph is an “ice cream cone” for $d = 2$) is not.

1.3.3 Operations that preserve convexity

There are two important operations that preserve convexity.

Lemma 1.9 (Exercise 4).

(i) Let $f_1, f_2, \ldots, f_m$ be convex functions, $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}^+$. Then $f := \sum_{i=1}^m \lambda_i f_i$ is convex on $\text{dom}(f) := \bigcap_{i=1}^m \text{dom}(f_i)$. 

(ii) Let $f$ be a convex function with $\text{dom}(f) \subseteq \mathbb{R}^d$, $g : \mathbb{R}^m \rightarrow \mathbb{R}^d$ an affine function, meaning that $g(x) = Ax + b$, for some matrix $A \in \mathbb{R}^{d \times m}$ and some vector $b \in \mathbb{R}^d$. Then the function $f \circ g$ (that maps $x$ to $f(Ax + b)$) is convex on $\text{dom}(f \circ g) := \{x \in \mathbb{R}^m : g(x) \in \text{dom}(f)\}$.

1.4 Minimizing convex functions

The main feature that makes convex functions attractive in optimization is that every local minimum is a global one, so we cannot “get stuck” in local optima. This is quite intuitive if we think of the graph of a convex function as being bowl-shaped.

Definition 1.10. A local minimum of $f : \text{dom}(f) \rightarrow \mathbb{R}$ is a point $x$ such that there exists $\varepsilon > 0$ with

$$f(x) \leq f(y) \quad \forall y \in \text{dom}(f) \text{ satisfying } \|y - x\| < \varepsilon.$$  

Lemma 1.11. Let $x^*$ be a local minimum of a convex function $f : \text{dom}(f) \rightarrow \mathbb{R}$. Then $x^*$ is a global minimum, meaning that

$$f(x^*) \leq f(y) \quad \forall y \in \text{dom}(f).$$

Proof. Suppose there exists $y \in \text{dom}(f)$ such that $f(y) < f(x^*)$ and define $y' := \lambda x^* + (1 - \lambda)y$ for $\lambda \in (0, 1)$. From convexity (1.1), we get that that $f(y') < f(x^*)$. Choosing $\lambda$ so close to 1 that $\|y' - x^*\| < \varepsilon$ yields a contradiction to $x^*$ being a local minimum. \qed

This does not mean that a convex function always has a global minimum. Think of $f(x) = x$ as a trivial example. But also if $f$ is bounded from below over $\text{dom}(f)$, it may fail to have a global minimum ($f(x) = e^x$). To ensure the existence of a global minimum, we need additional conditions. For example, it suffices if outside some ball $B$, all function values are larger than some value $f(x), x \in B$. In this case, we can restrict $f$ to $B$, without changing the smallest attainable value. And on $B$ (which is compact), $f$ attains a minimum by continuity (Lemma 1.6). An easy example: for $f(x_1, x_2) = x_1^2 + x_2^2$, we know that outside any ball containing 0, $f(x) > f(0) = 0$.

Another easy condition in the differentiable case is given by the following
Lemma 1.12. Suppose that $f$ is convex and differentiable over an open domain $\text{dom}(f)$. Let $x \in \text{dom}(f)$. If $\nabla f(x) = 0$, then $x$ is a global minimum.

Proof. Suppose that $\nabla f(x) = 0$. According to Lemma 1.7, we have

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) = f(x)$$

for all $y \in \text{dom}(f)$, so $x$ is a global minimum. \qed

The converse is also true and a corollary of Lemma 1.17 below [BV04, 4.2.3].

Lemma 1.13. Suppose that $f$ is convex and differentiable over an open domain $\text{dom}(f)$. Let $x \in \text{dom}(f)$. If $x$ is a global minimum then $\nabla f(x) = 0$.

1.4.1 Strictly convex functions

In general, a global minimum of a convex function is not unique (think of $f(x) = 0$ as a trivial example). However, if we forbid “flat” parts of the graph of $f$, a global minimum becomes unique (if it exists at all).

Definition 1.14 ([BV04, 3.1.1]). A function $f : \text{dom}(f) \to \mathbb{R}$ is strictly convex if (i) $\text{dom}(f)$ is convex and (ii) for all $x \neq y \in \text{dom}(f)$ and all $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

(1.4)

This means that the open line segment connecting $(x, f(x))$ and $(y, f(y))$ is pointwise strictly above the graph of $f$. For example, $f(x) = x^2$ is strictly convex. More generally (and following up on Lemma 1.8), if the Hessian $\nabla^2 f(x)$ is positive definite everywhere, then $f$ is strictly convex [BV04, 3.1.4]. The converse is false, though: $f(x) = x^4$ is strictly convex but has vanishing second derivative at $x = 0$.

Lemma 1.15. Let $f : \text{dom}(f) \to \mathbb{R}$ be strictly convex. Then $f$ has at most one global minimum.

Proof. Suppose $x^* \neq y^*$ are two global minima of value $f_{\text{min}}$, and let $z = \frac{1}{2}x^* + \frac{1}{2}y^*$. By (1.4),

$$f(z) < \frac{1}{2}f_{\text{min}} + \frac{1}{2}f_{\text{min}} = f_{\text{min}},$$

a contradiction to $x^*$ and $y^*$ being global minima. \qed
1.4.2 Example: Least squares

Suppose we want to fit a hyperplane to a set of data points \( x_1, \ldots, x_m \) in \( \mathbb{R}^d \), based on the hypothesis that the points actually come (approximately) from on a hyperplane. A classical method for this is least squares. For concreteness, let us do this in \( \mathbb{R}^2 \). Suppose that the data points are

\[(1, 10), (2, 11), (3, 11), (4, 10), (5, 9), (6, 10), (7, 9), (8, 10),\]

Also, for simplicity (and quite appropriately in this case), let us restrict to fitting a linear model, of more formally to fit non-vertical lines of the form \( y = w_0 + w_1 x \). If \( (x_i, y_i) \) is the \( i \)-th data point, the least squares fit chooses \( w_0, w_1 \) such that the least squares objective

\[ f(w_0, w_1) = \sum_{i=1}^{8} (w_1 x_i + w_0 - y_i)^2 \]

is minimized. It easily follows from Lemma 1.9 that \( f \) is convex. In fact,

\[ f(w_0, w_1) = 204 w_1^2 + 72 w_1 w_0 - 706 w_1 + 8 w_0^2 - 160 w_0 + 804, \quad (1.5) \]
so we can check convexity directly using the second order condition. We have gradient
\[ \nabla f(w_0, w_1) = (72w_1 + 16w_0 - 160, 408w_1 + 72w_0 - 706) \]
and Hessian
\[ \nabla^2 (w_0, w_1) = \begin{pmatrix} 16 & 72 \\ 72 & 408 \end{pmatrix}. \]
A 2 × 2 matrix is positive definite if the diagonal elements and the determinant are positive, which is the case here, so \( f \) is actually strictly convex and has a unique global minimum. To find it, we solve the linear system \( \nabla f(w_0, w_1) = (0, 0) \) of two equations in two unknowns and obtain the global minimum
\[ (w_0^*, w_1^*) = \left( \frac{43}{4}, -\frac{1}{6} \right). \]
Hence, the “optimal” line is
\[ y = -\frac{1}{6}x + \frac{43}{4}, \]
see Figure 1.6 (right).

1.4.3 Constrained Minimization

Frequently, we are interested in minimizing a convex function only over a subset \( X \) of its domain.

**Definition 1.16.** Let \( f : \text{dom}(f) \to \mathbb{R} \) be convex and let \( X \subseteq \text{dom}(f) \) be a convex set. \( x \in X \) is a minimizer of \( f \) over \( X \) if
\[ f(x) \leq f(y) \quad \forall y \in X. \]

If \( f \) is differentiable, minimizers of \( f \) over \( X \) have a very useful characterization.

**Lemma 1.17** ([BV04, 4.2.3]). Suppose that \( f \) is convex and differentiable over an open domain \( \text{dom}(f) \), and let \( X \subseteq \text{dom}(f) \) be a convex set. \( x^* \in X \) is a minimizer of \( f \) over \( X \) if and only if
\[ \nabla f(x^*)^\top (x - x^*) \geq 0 \quad \forall x \in X. \]
Geometrically, this means that $X$ is contained in the halfspace \( \{ x \in \mathbb{R}^d : \nabla f(x^*)^\top (x - x^*) \geq 0 \} \) (normal vector $\nabla f(x^*)$ pointing into the halfspace); see Figure 1.7. In still other words, $x - x^*$ forms a non-obtuse angle with $\nabla f(x^*)$ for all $x \in X$. Applying this with $X = \text{dom}(f)$, we recover Lemma 1.12 and its converse [BV04, 4.2.3].

![Figure 1.7: Optimality condition for constrained optimization](image)

We typically write constrained minimization problems in the form

\[
\text{argmin}\{f(x) : x \in X\}
\]

or

\[
\text{minimize } f(x) \\
\text{subject to } x \in X.
\]

1.5 Existence of a minimizer

The existence of a minimizer (or a global minimum if $X = \text{dom}(f)$) will be an assumption made by most minimization algorithms that we discuss later. In practice, such algorithms are being used (and often also work) if there is no minimizer. By “work”, we mean in this case that they compute a point $x$ such that $f(x)$ is close to $\inf_{y \in X} f(y)$, assuming that the infimum is finite (as in $f(x) = e^x$). But a sound theoretical analysis usually requires the existence of a minimizer. Therefore, this section develops tools that may help us in analyzing whether this is the case for a given convex function.
1.5.1 Sublevel sets and the Weierstrass Theorem

Definition 1.18. Let $f : \text{dom}(f) \to \mathbb{R}$, $\alpha \in \mathbb{R}$. The set

$$f^{\leq \alpha} := \{x \in \text{dom}(f) : f(x) \leq \alpha\}$$

is the $\alpha$-sublevel set of $f$; see Figure 1.8

![Figure 1.8: Sublevel set of a non-convex function (left) and a convex function (right)](image)

It is easy to see from the definition that every sublevel set of a convex function is convex. Moreover, as a consequence of continuity of $f$ (if $\text{dom}(f)$ is open), sublevel sets are closed. The following (known as the Weierstrass Theorem) just formalizes an argument that we have made earlier.

**Theorem 1.19.** Let $f : \text{dom}(f) \to \mathbb{R}$ be a convex function, $\text{dom}(f)$ open, and suppose there is a nonempty and bounded sublevel set $f^{\leq \alpha}$. Then $f$ has a global minimum.

**Proof.** We know that $f$—as a continuous function over $\text{dom}(f)$—attains a minimum over the closed and bounded (= compact) set $f^{\leq \alpha} \subseteq \text{dom}(f)$ at some $x^*$. This $x^*$ is also a global minimum as it has value $f(x^*) \leq \alpha$, while any $x \notin f^{\leq \alpha}$ has value $f(x) > \alpha$. \qed

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1.5.2 Recession cone and lineality space

What happens if there is no bounded level set? Then $f$ may ($f(x) = 0$) or may not ($f(x) = x, f(x) = e^x$) have a global minimum. But what cannot happen is that some nonempty level sets are bounded, and others aren’t. Also, in the unbounded case, all nonempty level sets have the same “reason” for being unbounded, namely the same directions of recession. In the remainder of this section, we assume for simplicity that $\text{dom}(f) = \mathbb{R}^d$, and we are only considering unconstrained minimization. But most of the material can be adapted to general domains and to constrained optimization over a convex $X \subseteq \text{dom}(f)$. We closely follow Bertsekas [Ber05].

Recession cone and lineality space of a convex set.

Definition 1.20. Let $C \subseteq \mathbb{R}^d$ be a convex set. $y \in \mathbb{R}^d$ is a direction of recession of $C$ if for some $x \in C$ and all $\lambda \geq 0$, $x + \lambda y \in C$.

This means that $C$ is unbounded in direction $y$. Whether $y$ is a direction of recession only depends on $C$ and not on the particular $x$, assuming that $C$ is closed (otherwise, it may be false).

Lemma 1.21. Let $C \subseteq \mathbb{R}^d$ be a nonempty closed convex set, $y \in \mathbb{R}^d$. The following statements are equivalent.

(i) $\exists x \in C : x + \lambda y \in C$ for all $\lambda \geq 0$.

(ii) $\forall x \in C : x + \lambda y \in C$ for all $\lambda \geq 0$.

Proof. We need to show that (i) implies (ii), so choose $x \in C, y \in \mathbb{R}^d$ such that $x + \lambda y \in C$ for all $\lambda \geq 0$. Fix $\lambda > 0$, let $z = \lambda y$ and $x' \in C$. To get (ii), we prove that $x' + z \in C$. To this end, we define sequences $(w_k), (z_k), k \in \mathbb{N}$ via

$$w_k := x + k z \in C \quad \text{(by (i))}$$
$$z_k := \frac{1}{k}(w_k - x') = z + \frac{1}{k}(x - x'),$$

see Figure 1.9. By definition of a convex set, we have $x' + z_k = \frac{1}{k}w_k + (1 - \frac{1}{k})x' \in C$. Moreover, $z_k$ converges to $z$, so $x' + z_k$ converges to $x' + z \in C$, and this is an element of $C$, since $C$ is closed. \qed
The directions of recession of $C$ actually form a convex cone, a set that is closed under taking nonnegative linear combinations. This is known as the recession cone $R(C)$ of $C$; see Figure 1.10 (left).

**Lemma 1.22.** Let $C \subseteq \mathbb{R}^d$ be a closed convex set, and let $y_1, y_2$ be directions of recession of $C$; $\lambda_1, \lambda_2 \in \mathbb{R}_+$. Then $y = \lambda_1 y_1 + \lambda_2 y_2$ is also a direction of recession of $C$.

**Proof.** The statement is trivial if $y = 0$. Otherwise, after scaling $y$ by $1/(\lambda_1 + \lambda_2) > 0$ (which does not affect the property of being a direction of recession), we may assume that $\lambda_1 + \lambda_2 = 1$. Now, for all $x \in C$ and all $\lambda \in \mathbb{R}$, we get

$$x + \lambda y = x + \lambda_1 \lambda y_1 + \lambda_2 \lambda y_2 = \lambda_1 (\underbrace{x + \lambda y_1}_{\in C}) + \lambda_2 (\underbrace{x + \lambda y_2}_{\in C}) \in C.$$ 

\[\square\]
Definition 1.23. Let $C \subseteq \mathbb{R}^d$ be a convex set. $y \in \mathbb{R}^d$ is a direction of constancy of $C$ if both $y$ and $-y$ are directions of recession of $C$.

This means that $C$ is unbounded along the whole line spanned by $y$. The directions of constancy form a linear subspace, the lineality space $L(C)$ of $C$; see Figure 1.10 (right).

Lemma 1.24. Let $C \subseteq \mathbb{R}^d$ be a closed convex set, and let $y_1, y_2$ be directions of constancy of $C$; $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $y = \lambda_1 y_1 + \lambda_2 y_2$ is also a direction of constancy of $C$.

Proof. After replacing $y_i$ with the direction of recession $-y_i$ if necessary ($i = 1, 2$), we may assume that $\lambda_1, \lambda_2 \geq 0$, so $y$ is a direction of recession by Lemma 1.22. The same argument works for $-y$, so $y$ is a direction of constancy.

Recession cone and lineality space of a convex function. For this, we look at directions of recession of sublevel sets (which are closed and convex in our case).

Lemma 1.25. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Any two nonempty level sets $f \leq \alpha, f \leq \alpha'$ have the same recession cones, i.e. $R(f \leq \alpha) = R(f \leq \alpha')$.

Proof. Let $y$ be a direction of recession for $f \leq \alpha$, i.e. for all $x \in f \leq \alpha$ and all $\lambda \geq 0$, we have

$$f(x + \lambda y) \leq \alpha.$$  

We claim that this implies the stronger bound

$$f(x + \lambda y) \leq f(x). \tag{1.8}$$

In words, $f$ is nonincreasing along any direction of recession. Using this, the statement follows: Among the $x \in f \leq \alpha$, there exists $x' \in f \leq \alpha'$, and then we have $f(x' + \lambda y) \leq f(x') \leq \alpha'$, so $y$ is a direction of recession for $f \leq \alpha'$.

To prove (1.8), we fix $\lambda$ and let $z = \lambda y$. With $w_k := x + k z \in f \leq \alpha$, we have

$$x + z = \left(1 - \frac{1}{k}\right)x + \frac{1}{k}w_k,$$

so convexity of $f$ yields

$$f(x + z) \leq \left(1 - \frac{1}{k}\right)f(x) + \frac{1}{k}f(w_k) \leq \left(1 - \frac{1}{k}\right)f(x) + \frac{1}{k}\alpha \to k \to \infty f(x).$$

Hence, $f(x + z) \leq f(x)$; see Figure 1.11. \hfill \Box
Figure 1.11: The proof of (1.8)

**Definition 1.26.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex. $y \in \mathbb{R}^d$ is a direction of recession (of constancy, respectively) of $f$ if $y$ is a direction of recession (of constancy, respectively) for some (equivalently, for every) nonempty sublevel set. The set of directions of recessions of $f$ is called the recession cone $R(f)$ of $f$. The set of directions of constancy of $f$ is called the lineality space $L(f)$ of $f$.

We can characterize recession cone and lineality space of $f$ directly, without looking at sublevel sets (the proof is Exercise 5). The conditions of Lemma 1.27(ii) and Lemma 1.28(ii) finally explain the terms “direction of recession” and “direction of constancy”.

**Lemma 1.27.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex. The following statements are equivalent.

(i) $y \in \mathbb{R}^d$ is a direction of recession of $f$.

(ii) $f(x + \lambda y) \leq f(x)$ for all $x \in \mathbb{R}^d$ and all $\lambda \in \mathbb{R}_+$.

(iii) $(y, 0)$ is a ("horizontal") direction of recession of (the closed convex set) $\text{epi}(f)$.

**Lemma 1.28.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex. The following statements are equivalent.

(i) $y \in \mathbb{R}^d$ is a direction of constancy of $f$.

(ii) $f(x + \lambda y) = f(x)$ for all $x \in \mathbb{R}^d$ and all $\lambda \in \mathbb{R}$.

(iii) $(y, 0)$ is a ("horizontal") direction of constancy of (the closed convex set) $\text{epi}(f)$.
1.5.3 Coercive convex functions

Definition 1.29. A convex function $f$ is coercive if its recession cone is trivial, meaning that 0 is its only direction of recession.\(^1\)

Coercivity means that along any direction, $f(x)$ goes to infinity. An example of a coercive convex function is $f(x_1, x_2) = x_1^2 + x_2^2$. Non-coercive functions are $f(x) = x$ and $f(x) = e^x$ (any $y \leq 0$ is a direction of recession). For a constant function $f : \mathbb{R}^d \to \mathbb{R}$, every direction $y$ is a direction of recession. In general, affine functions are never coercive.

Lemma 1.30. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a coercive convex function. Then every sublevel set $f \leq \alpha$ is bounded.

This may seem obvious, as $f \leq \alpha$ is bounded in every direction by coercivity. But we still need an argument that there is a global bound.

Proof. We prove that $f \leq \alpha$ is bounded for $\alpha > f(0)$; since $f \leq \alpha' \subseteq f \leq \alpha$ if $\alpha' < \alpha$, the result extends to all sublevel sets.

Let $S^{d-1} = \{y \in \mathbb{R}^d : \|y\| = 1\}$ be the unit sphere. We define a function $g : S^{d-1} \to \mathbb{R}$ via

$$g(y) = \max\{\lambda \geq 0 : f(\lambda y) \leq \alpha\}, \quad y \in S^{d-1}.$$ 

Since $f$ is continuous and has no nonzero directions of recession, the set $\{\lambda \geq 0 : f(\lambda y) \leq \alpha\}$ is closed and bounded (it is actually an interval, by convexity of $f$), so the maximum exists and $g(y)$ is well-defined. We claim that $g$ is continuous.

Let $(y_k)_{k \in \mathbb{N}}$ be a sequence of unit vectors such that $\lim_{k \to \infty} y_k = y \in S^{d-1}$. We need to show that $\lim_{k \to \infty} g(y_k') = g(y)$. Let us fix $\varepsilon > 0$ arbitrarily small. For $\lambda := g(y) - \varepsilon \geq 0$, we have $f(\lambda y) < \alpha$ (an easy consequence of convexity of $f$ and $\alpha > f(0)$). And for $\overline{\lambda} := g(y) + \varepsilon$, we get $f(\overline{\lambda} y) > \alpha$ by definition of $g(y)$. Continuity of $f$ then yields $\lim_{k \to \infty} f(\lambda y_k) = f(\lambda y) < \alpha$ and $\lim_{k \to \infty} f(\overline{\lambda} y_k) = f(\overline{\lambda} y) > \alpha$. Hence, for sufficiently large $k$, $g(y_k) \in (\lambda, \overline{\lambda}) = (g(y) - \varepsilon, g(y) + \varepsilon)$, and $\lim_{k \to \infty} g(y_k') = g(y)$ follows.

As a continuous function, $g$ attains a maximum $\lambda^*$ over the compact set $S^{d-1}$, and this means that $f \leq \alpha$ is contained in the closed $\lambda^*$-ball around the origin. Hence, $f \leq \alpha$ is bounded. \(\square\)

\(^1\)The usual definition of a coercive function is that $f(x) \to \infty$ whenever $\|x\| \to \infty$. In the convex case, both definitions agree.
Together with Theorem 1.19, we obtain

**Theorem 1.31.** Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a coercive convex function. Then $f$ has a global minimum.

### 1.5.4 Weakly coercive convex functions

It turns out that we can allow nontrivial directions of recession and still guarantee a global minimum.

**Definition 1.32.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex. $f$ is called weakly coercive if its recession cone equals its lineality space, i.e. every direction of recession is a direction of constancy.

$f(x) = 0$ is a trivial example of a non-coercive but weakly coercive function. A more interesting example is $f(x_1, x_2) = x_2^3$. Here, the directions of recession are all vectors of the form $y = (0, x_2)$, and these are at the same time directions of constancy.

**Theorem 1.33.** Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a weakly coercive convex function. Then $f$ has a global minimum.

**Proof.** We know that the lineality space $L$ of $f$ is a linear subspace of $\mathbb{R}^d$: by Definition 1.26, $L$ is the lineality space of every nonempty (closed and convex) sublevel set, and as such it is closed under taking linear combinations (Lemma 1.24). Let $L^\perp$ be the orthogonal complement of $L$. Restricted to $L^\perp$, $f$ is coercive, as $L^\perp$ is orthogonal to any direction of constancy, equivalently to every direction of recession, since $f$ is weakly coercive. $L^\perp$ can therefore contain only the trivial direction of recession. It follows that $f|_{L^\perp}$ has a global minimum $x^* \in L^\perp$ by Theorem 1.31 (which we can apply after identifying $L^\perp$ w.l.o.g. with $\mathbb{R}^m$ for some $m \leq n$). This is also a global minimum of $f$. To see this, let $z \in \mathbb{R}^d$ and write it (uniquely) in the form $z = x + y$ with $x \in L^\perp$ and $y \in L$. Then we get

$$f(z) = f(x + y) = f(x) \geq f(x^*),$$

where the second equality follows from $y$ being a direction of constancy; see Lemma 1.28(ii).
1.6 Examples

In the following two sections, we give two examples of convex function minimization tasks that arise from machine learning applications.

1.6.1 Handwritten digit recognition

Suppose you want to write a program that recognizes handwritten decimal digits $0, 1, \ldots, 9$. You have a set $P$ of grayscale images ($28 \times 28$ pixels, say) that represent handwritten decimal digits, and for each image $x \in P$, you know the digit $d(x) \in \{0, \ldots, 9\}$ that it represents, see Figure 1.12. You want to train your program with the set $P$, and after that use it to recognize handwritten digits in arbitrary $28 \times 28$ images.

![Figure 1.12: Some training images from the MNIST data set (picture from http://corochann.com/mnist-dataset-introduction-1138.html)](image)

The classical approach is the following. We represent an image as a feature vector $x \in \mathbb{R}^{784}$, where $x_i$ is the gray value of the $i$-th pixel (in some order). During the training phase, we compute a matrix $W \in \mathbb{R}^{10 \times 784}$ and
then use the vector \( y = W x \in \mathbb{R}^{10} \) to predict the digit seen in an arbitrary image \( x \). The idea is that \( y_j, j = 0, \ldots, 9 \) corresponds to the probability of the digit being \( j \). This doesn’t work directly, since the entries of \( y \) may be negative and generally do not sum up to 1. But we can convert \( y \) to a vector \( z \) of actual probabilities, such that a small \( y_j \) leads to a small probability \( z_j \) and a large \( y_j \) to a large probability \( z_j \). How to do this is not canonical, but here is a well-known formula that works:

\[
  z_j = z_j(y) = \frac{e^{y_j}}{\sum_{k=0}^{9} e^{y_k}}. \quad (1.9)
\]

The classification then simply outputs digit \( j \) with probability \( z_j \). The matrix \( W \) is chosen such that it (approximately) minimizes the classification error on the training set \( P \). Again, it is not canonical how we measure classification error; here we use the following loss function to evaluate the error induced by a given matrix \( W \).

\[
  \ell(W) = -\sum_{x \in P} \ln z_{d(x)}(W x) = \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(W x)_j} \right) - (W x)_{d(x)} \right). \quad (1.10)
\]

This function “punishes” images for which the correct digit \( j \) has low probability \( z_j \) (corresponding to a significantly negative value of \( \log z_j \)). In an ideal world, the correct digit would always have probability 1, resulting in \( \ell(W) = 0 \). But under (1.9), probabilities are always strictly between 0 and 1, so we have \( \ell(W) > 0 \) for all \( W \).

Exercise 6 asks you to prove that \( \ell \) is convex. In Exercise 7, you will characterize the situations in which \( \ell \) has a global minimum.

### 1.6.2 Master’s Admission

The computer science department of a well known Swiss university is admitting top international students to its MSc program, in a competitive application process. Applicants are submitting various documents (GPA, TOEFL test scores, GRE test scores, reference letters, . . .). During the evaluation of an application, the admission committee would like to compute a (rough) forecast of the applicant’s performance in the MSc program, based on the submitted documents.\(^2\)

\(^2\)Any resemblance to real departments is purely coincidental. Also, no serious department will base performance forecasts on data from 10 students, as we will do it here.
Data on the actual performance of admitted students is available. To keep things simple in the following example, let’s base the forecast on GPA (grade point average) and TOEFL (Test of English as a Foreign Language) only. GPA scores are normalized to a scale with a minimum of 0.0 and a maximum of 4.0, where admission starts from 3.5. TOEFL scores are on an integer scale between 0 and 120, where admission starts from 100.

Table 1.1 contains the known data. GGPA (graduation grade point average on a Swiss grading scale) is the average grade obtained by an admitted student over all courses in the MSc program. The Swiss scale goes from 1 to 6 where 1 is the lowest grade, 6 is the highest, and 4 is the lowest passing grade.

<table>
<thead>
<tr>
<th>GPA</th>
<th>TOEFL</th>
<th>GGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.52</td>
<td>100</td>
<td>3.92</td>
</tr>
<tr>
<td>3.66</td>
<td>109</td>
<td>4.34</td>
</tr>
<tr>
<td>3.76</td>
<td>113</td>
<td>4.80</td>
</tr>
<tr>
<td>3.74</td>
<td>100</td>
<td>4.67</td>
</tr>
<tr>
<td>3.93</td>
<td>100</td>
<td>5.52</td>
</tr>
<tr>
<td>3.88</td>
<td>115</td>
<td>5.44</td>
</tr>
<tr>
<td>3.77</td>
<td>115</td>
<td>5.04</td>
</tr>
<tr>
<td>3.66</td>
<td>107</td>
<td>4.73</td>
</tr>
<tr>
<td>3.87</td>
<td>106</td>
<td>5.03</td>
</tr>
<tr>
<td>3.84</td>
<td>107</td>
<td>5.06</td>
</tr>
</tbody>
</table>

Table 1.1: Data for 10 admitted students: GPA and TOEFL scores (at time of application), GGPA (at time of graduation)

As in Section 1.4.2, we are attempting a linear regression with least squares fit, i.e. we are making the hypothesis that

\[ \text{GGPA} \approx w_0 + w_1 \cdot \text{GPA} + w_2 \cdot \text{TOEFL}. \]  

However, in our scenario, the GPA scores span a range of only 0.5 while the TOEFL scores span a range of 20. The resulting least squares objective would be somewhat ugly; we already saw this in our previous example (1.5), where the data points had large second coordinate, resulting in the \( w_1 \)-scale being very different from the \( w_2 \)-scale. This time, we normalize first.
The general setting is this: we have \( n \) inputs \( x_1, \ldots, x_n \), where each vector \( x_i \in \mathbb{R}^d \) consists of \( d \) input variables; then we have \( n \) outputs \( y_1, \ldots, y_n \in \mathbb{R} \). Each pair \((x_i, y_i)\) is an observation. In our case, \( d = 2, n = 10 \), and for example, \(((3.52, 100), 3.92)\) is an observation (of a student failing, unfortunately).

We first want to assume that the inputs and outputs are centered, meaning that
\[
\sum_{i=1}^{n} x_i = 0, \quad \sum_{i=1}^{n} y_i = 0.
\]

This can be achieved by simply subtracting the mean \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) from every input and the mean \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \) from every output. In our example, this yields the numbers in Table 1.2 (left).

<table>
<thead>
<tr>
<th>GPA</th>
<th>TOEFL</th>
<th>GGPA</th>
<th>GPA</th>
<th>TOEFL</th>
<th>GGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.24</td>
<td>-7.2</td>
<td>-0.94</td>
<td>-2.04</td>
<td>-1.28</td>
<td>-0.94</td>
</tr>
<tr>
<td>-0.10</td>
<td>1.8</td>
<td>-0.52</td>
<td>-0.88</td>
<td>0.32</td>
<td>-0.52</td>
</tr>
<tr>
<td>-0.01</td>
<td>5.8</td>
<td>-0.05</td>
<td>-0.05</td>
<td>1.03</td>
<td>-0.05</td>
</tr>
<tr>
<td>-0.02</td>
<td>-7.2</td>
<td>-0.18</td>
<td>-0.16</td>
<td>-1.28</td>
<td>-0.18</td>
</tr>
<tr>
<td>0.17</td>
<td>-7.2</td>
<td>0.67</td>
<td>1.42</td>
<td>-1.28</td>
<td>0.67</td>
</tr>
<tr>
<td>0.12</td>
<td>7.8</td>
<td>0.59</td>
<td>1.02</td>
<td>1.39</td>
<td>0.59</td>
</tr>
<tr>
<td>0.01</td>
<td>7.8</td>
<td>0.19</td>
<td>0.06</td>
<td>1.39</td>
<td>0.19</td>
</tr>
<tr>
<td>-0.10</td>
<td>-0.2</td>
<td>-0.12</td>
<td>-0.88</td>
<td>-0.04</td>
<td>-0.12</td>
</tr>
<tr>
<td>0.11</td>
<td>-1.2</td>
<td>0.17</td>
<td>0.89</td>
<td>-0.21</td>
<td>0.17</td>
</tr>
<tr>
<td>0.07</td>
<td>-0.2</td>
<td>0.21</td>
<td>0.62</td>
<td>-0.04</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Table 1.2: Centered observations (left); normalized inputs (right)

Centering has the effect that the global minimum \( w^* \) of the least squares objective
\[
f(w^*) = \sum_{i=1}^{n} (w_0 + \sum_{j=1}^{d} w_j x_{ij} - y_i)^2
\]
satisfies \( w^*_0 = 0 \) (Exercise 10), so that we can simply omit this variable in the sequel.

Finally, we assume that all \( d \) input variables are on the same scale, meaning that
\[
\frac{1}{n} \sum_{i=1}^{n} x_{ij}^2 = 1, \quad j = 1, \ldots, d. \]
To achieve this, we simply need to multiply $x_{ij}$ by $\sqrt{\frac{n}{\sum_{i=1}^{d} x_{ij}^2}}$ for all $i, j$. For our data set, the resulting normalized data are shown in Table 1.2 (right). Now the least squares objective (after omitting $w_0$) is

$$f(w_1, w_2) = \sum_{i=1}^{10} (w_1 x_{i1} + w_2 x_{i2} - y_i)^2$$

$$\approx 10w_1^2 + 10w_2^2 + 1.99w_1w_2 - 8.7w_1 - 2.79w_2 + 2.09.$$  

This is minimized at

$$w^* = (w_1^*, w_2^*) \approx (0.43, 0.097),$$

so if our initial hypothesis (1.11) is true, we should have

$$y_i \approx y_i^* = 0.43x_{i1} + 0.097x_{i2} \quad (1.12)$$

in the normalized data. This can quickly be checked, and the results aren’t perfect, but not too bad, either; see Table 1.3 (ignore the last column for now).

<table>
<thead>
<tr>
<th>$x_{i1}$</th>
<th>$x_{i2}$</th>
<th>$y_i$</th>
<th>$y_i^*$</th>
<th>$z_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.04</td>
<td>-1.28</td>
<td>-0.94</td>
<td>-1.00</td>
<td>-0.87</td>
</tr>
<tr>
<td>-0.88</td>
<td>0.32</td>
<td>-0.52</td>
<td>-0.35</td>
<td>-0.37</td>
</tr>
<tr>
<td>-0.05</td>
<td>1.03</td>
<td>-0.05</td>
<td>0.08</td>
<td>-0.02</td>
</tr>
<tr>
<td>-0.16</td>
<td>-1.28</td>
<td>-0.18</td>
<td>-0.19</td>
<td>-0.07</td>
</tr>
<tr>
<td>1.42</td>
<td>-1.28</td>
<td>0.67</td>
<td>0.49</td>
<td>0.61</td>
</tr>
<tr>
<td>1.02</td>
<td>1.39</td>
<td>0.59</td>
<td>0.57</td>
<td>0.44</td>
</tr>
<tr>
<td>0.06</td>
<td>1.39</td>
<td>0.19</td>
<td>0.16</td>
<td>0.03</td>
</tr>
<tr>
<td>-0.88</td>
<td>-0.04</td>
<td>-0.12</td>
<td>-0.38</td>
<td>-0.37</td>
</tr>
<tr>
<td>0.89</td>
<td>-0.21</td>
<td>0.17</td>
<td>0.36</td>
<td>0.38</td>
</tr>
<tr>
<td>0.62</td>
<td>-0.04</td>
<td>0.21</td>
<td>0.26</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Table 1.3: Outputs $y_i^*$ predicted by the linear model (1.12) and by the model $z_i^* = 0.43x_{i1}$ that simply ignores the second input variable

What we also see from (1.12) is that the first input variable (GPA) has a much higher influence on the output (GGPA) than the second one (TOEFL).
In fact, if we drop the second one altogether, we obtain outputs $z^\star_i$ (last column in Table 1.3) that seem equivalent to the predicted outputs $y^\star_i$ within the level of noise that we have anyway.

We conclude that TOEFL scores are probably not indicative for the performance of admitted students, so the admission committee should not care too much about them. Requiring a minimum score of 100 might make sense, but whenever an applicant reaches at least this score, the actual value does not matter.

The LASSO. So far, we have computed linear functions $y = 0.43x_1 + 0.097x_2$ and $z = 0.43x_1$ that “explain” the historical data from Table 1.1. If we believe that hypothesis (1.11) also holds for future applicants, we can indeed use these functions to make performance forecasts. Using $z$ instead of $y$ introduces a bias, though. For example, if the average TOEFL score of future applicants is above 110, $z$ will underestimate performance by ignoring the (small) positive effect of a high TOEFL score. The advantages of $z$ over $y$ are (i) better interpretability ($z$ “knows” that TOEFL is non-indicative, while $y$ doesn’t); and (ii) possibly more stable estimates (variance caused by a non-indicative input variable is removed).

The question is: how can we in general improve the quality of our forecast in the presence of non-indicative input variables such as the TOEFL score in our example? We can (as done above) just forget about a variable of weight close to 0 in the least squares solution. However, for this, we need to define what it means to be close to 0; and it may happen that small changes in the data lead to different variables being dropped if their weights are around the threshold. A more elegant solution has been suggested by Tibshirani in 1996 [Tib96]. Instead of minimizing the least squares objective globally, it is minimized over a suitable $\ell_1$-ball (ball in the 1-norm):

$$
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^n \| w^T x_i - y_i \|^2 \\
\text{subject to} & \quad \|w\|_1 \leq R,
\end{align*}
$$

(1.13)

where $R \in \mathbb{R}_+$ is some parameter. Here $w \in \mathbb{R}^d$ is the vector of optimization variables. In our case, $w = (w_1, w_2)$, and if we for example

$$
\begin{align*}
\text{minimize} & \quad f(w_1, w_2) = 10w_1^2 + 10w_2^2 + 1.99w_1w_2 - 8.7w_1 - 2.79w_2 + 2.09 \\
\text{subject to} & \quad |w_1| + |w_2| \leq 0.2,
\end{align*}
$$

(1.14)
we obtain $\mathbf{w}^* = (w_1^*, w_2^*) = (0.2, 0)$: the non-indicative TOEFL score has disappeared automatically! For $R = 0.3$, the same happens (with $w_1^* = 0.3$, respectively). For $R = 0.4$, the TOEFL score starts creeping back in: we get $(w_1^*, w_2^*) \approx (0.36, 0.036)$. For $R = 0.5$, we have $(w_1^*, w_2^*) \approx (0.41, 0.086)$, while for $R = 0.6$ (and all larger values of $R$), we recover the original solution $(w_1^*, w_2^*) = (0.43, 0.097)$.

This phenomenon is not restricted to $d = 2$. The constrained minimization problem (1.13) is called the LASSO (least absolute shrinkage and selection operator) and has the tendency to assign weight 0 to and thus remove non-indicative input variables, where $R$ controls how aggressive the selection is, and how much bias it potentially introduces (the smaller the value $R$, the more aggressive the selection, and the higher the bias).

In our example, it’s easy to get an intuition why this works. Let’s look at the case $R = 0.2$. The smallest value attainable in (1.14) is the smallest $\alpha$ such that that the (elliptical) sublevel set $f_{\leq \alpha}$ of the least squares objective $f$ still intersects the $\ell_1$-ball $\{(w_1, w_2) : |w_1| + |w_2| \leq 0.2 \}$. This smallest value turns out to be $\alpha = 0.75$, see Figure 1.13. For this value of $\alpha$, the sublevel set intersects the $\ell_1$-ball exactly in one point, namely $(0.2, 0)$.

\begin{align*}
10w_1^2 + 10w_2^2 + 1.99w_1w_2 - 8.7w_1 - 2.79w_2 + 2.09 &= 0.75
\end{align*}

At $(0.2, 0)$, the ellipse $\{(w_1, w_2) : f(w_1, w_2) = \alpha \}$ is “vertical enough”
to just intersect the corner of the $\ell_1$-ball. The reason is that the center of the ellipse is relatively close to the $w_1$-axis, when compared to its size. As $R$ increases, the relevant value of $\alpha$ decreases, the ellipse gets smaller and less vertical around the $w_1$-axis; until it eventually stops intersecting the $\ell_1$-ball \{(w_1, w_2) : |w_1| + |w_2| \leq R\} in a corner (dashed situation in Figure 1.13, for $R = 0.4$).

Even though we have presented a toy example in this section, the back-ground is real. The theory of admission and in particular performance forecasts has been developed in a recent PhD thesis by Zimmermann [Zim16].

### 1.7 Exercises

**Exercise 1.** Prove Jensen’s inequality (Lemma 1.5)!

**Solution:** For $m = 1$, there is nothing to prove, and for $m = 2$, the statement holds by convexity of $f$. For $m > 2$, we proceed by induction. If $\lambda_m = 1$ (and hence all other $\lambda_i$ are zero), the statement is trivial. Otherwise, let $x = \sum_{i=1}^{m} \lambda_i x_i$ and define

$$y = \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} x_i.$$  

Thus we have $x = (1 - \lambda_m)y + \lambda_m x_m$. Also observe that $\sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} = 1$.

By convexity and Jensen’s inequality that we inductively assume to hold for $m - 1$ terms, we get

$$f(x) = f((1 - \lambda_m)y + \lambda_m x_m) \leq (1 - \lambda) f \left( \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} x_i \right) + \lambda_m f(x_m) \leq (1 - \lambda) \left( \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m} f(x_i) \right) + \lambda_m f(x_m) = \sum_{i=1}^{m} \lambda_i f(x_i).$$

**Exercise 2.** Prove that a convex function is continuous (Lemma 1.6)!

**Hint:** First prove that a convex function $f$ is bounded on any cube $C = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_d, u_d] \subseteq \text{dom}(f)$, with the maximum value occurring on some corner of the cube (a point $z$ such that $z_i \in \{l_i, u_i\}$ for all $i$). Then use this fact to show that—given $x \in \text{dom}(f)$, $\varepsilon > 0$—all $y$ in a sufficiently small cube around $x$ satisfy $|f(y) - f(x)| < \varepsilon$. 

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Solution: For a cube $C$, we let $e(C) = \max\{i : l_i < u_i\}$ be its extent. If $l_i = u_i$ for all $i$, we define $e(C) = 0$. A cube of extent 0 is therefore a single point, so $f$ is bounded over any cube of extent 0, with the maximum value occurring on the single corner $(l_0, \ldots, l_d)$. Now assume that $f$ is bounded over any cube of extent at most $m - 1$, with the maximum value occurring on a corner. Let $C$ be a cube of extent $m$ and $x \in C$.

Define $\underline{x} = x - (x_m - l_m) e_m$ and $\overline{x} = x + (u_m - x_m) e_m$, where $e_m$ is the $m$-th unit vector. In other words, $\underline{x}$ results from $x$ by replacing the $m$-th coordinate with $l_m$, and $\overline{x}$ results from $x$ by replacing the $m$-th coordinate with $u_m$. With $\lambda = (u_m - x_m)/(u_m - l_m)$, we then have

$$x = \lambda \underline{x} + (1 - \lambda) \overline{x},$$

so convexity yields

$$f(x) \leq \lambda f(\underline{x}) + (1 - \lambda) f(\overline{x}).$$

Moreover, $\underline{x}$ is in the “lower” subcube $\underline{C} = C \cap \{z \in \mathbb{R}^d : z_m = l_m\}$ of extent at most $m - 1$, and $\overline{x}$ in the “upper” subcube $\overline{C} = C \cap \{z \in \mathbb{R}^d : z_m = u_m\}$ of extent at most $m - 1$. Let $\underline{\alpha}$ be the maximum value of $f$ over $\underline{C}$, attained at some corner, and $\overline{\alpha}$ the maximum value of $f$ over $\overline{C}$, again attained at some corner. Then we further get

$$f(x) \leq \lambda \underline{\alpha} + (1 - \lambda) \overline{\alpha} \leq \alpha := \max(\underline{\alpha}, \overline{\alpha}).$$

Hence, $\alpha$ (attained at some corner of $C$ whose corners are just the ones of $\underline{C}$ and $\overline{C}$ together) is the maximum value of $f$ over $C$.

Now we can move on to continuity of $f$. Let $x \in \text{dom}(f)$ be fixed, and let $C \subseteq \text{dom}(f)$ be some cube of extent $d$ containing $x$ in its interior (such a cube exists as $\text{dom}(f)$ was assumed to be open). W.l.o.g. assume that $\max\{f(z) : z \in C\} = 1$. For a given $y \in C$ (also in the interior), we now define two values

$$\underline{\lambda} = \max\{\lambda \in \mathbb{R}^+ : x + \lambda(y - x) \in C\} > 1,$$

$$\overline{\lambda} = \max\{\lambda \in \mathbb{R}^+ : y + \lambda(x - y) \in C\} > 1.$$ 

and corresponding points

$$\underline{y} = x + \underline{\lambda}(y - x),$$

$$\overline{y} = y + \overline{\lambda}(x - y)$$

on the boundary of $C$. 

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We also get
\[ y = \frac{1}{\lambda}y + \frac{\lambda - 1}{\lambda}x, \]
\[ x = \frac{1}{\lambda}y + \frac{\lambda - 1}{\lambda}y. \]

Hence, by convexity (and using that the maximum value of \( f \) over \( C \) is 1),
\[ f(y) \leq \frac{1}{\lambda}f(y) + \frac{\lambda - 1}{\lambda}f(x) \leq \frac{1}{\lambda} + f(x), \]
\[ f(x) \leq \frac{1}{\lambda}f(y) + \frac{\lambda - 1}{\lambda}f(y) \leq \frac{1}{\lambda} + f(y). \]

Thus, to get \(|f(x) - f(y)| < \varepsilon\), we need to ensure that \( \lambda, \Delta > 1/\varepsilon \). Let \( r \) denote the distance of \( x \) to the boundary of \( C \). Then we get
\[ r \leq \|y - x\| = \|\lambda(y - x)\| = \lambda\|x - y\|, \]
\[ r \leq \|y - x\| \leq \|y - y\| = \|\Delta(x - y)\| = \Delta\|y - x\|. \]

Hence, if \( \|x - y\| < r\varepsilon \), then \( \lambda, \Delta > 1/\varepsilon \) and \( |f(x) - f(y)| < \varepsilon \), so continuity holds.

**Exercise 3.** Prove that the function \( d_y : \mathbb{R}^d \to \mathbb{R}, \ x \mapsto \|x - y\|^2 \) is strictly convex for any \( y \in \mathbb{R}^d \). (This is essentially an exercise in computing gradients and Hessians.)
Solution: By Lemma 1.8, it suffices to show that $\nabla^2 d_y(x)$ is positive definite for every $x \in \mathbb{R}^d$. We compute

$$d_y(x) = (x - y)^T(x - y), \quad \nabla d_y(x) = 2(x - y), \quad \nabla^2 d_y(x) = 2I \succ 0,$$

where $I$ denotes the identity matrix. The claim follows.

Exercise 4. Prove Lemma 1.9! Can (ii) be generalized to show that for two convex functions $f, g$, the function $f \circ g$ is convex as well?

Solution:

(i) Let $x, y \in \text{dom}(f)$ and $\lambda \in [0, 1]$ be arbitrary. We simply compute

$$f(\lambda x + (1 - \lambda)y) = \sum_{i=1}^{m} \lambda_i f_i(\lambda x + (1 - \lambda)y) \leq \sum_{i=1}^{m} \lambda_i (\lambda f_i(x) + (1 - \lambda) f_i(y)) = \lambda \cdot \sum_{i=1}^{m} \lambda_i f_i(x) + (1 - \lambda) \cdot \sum_{i=1}^{m} \lambda_i f_i(y),$$

where the inequality makes use of convexity of the individual $f_i$ and of the fact that the $\lambda_i$ are non-negative.

(ii) Let $x, y \in \text{dom}(f \circ g)$ and $\lambda \in [0, 1]$ be arbitrary. We simply compute

$$(f \circ g)(\lambda x + (1 - \lambda)y) = f(A(\lambda x + (1 - \lambda)y) + b) = f(\lambda \cdot (Ax + b) + (1 - \lambda) \cdot (Ay + b)) \leq \lambda \cdot f(Ax + b) + (1 - \lambda) \cdot f(Ay + b),$$

where the inequality makes use of convexity of $f$ and of the fact that both $g(x) = Ax + b$ and $g(y) = Ay + b$ are in the domain of $f$.

If two functions $f$ and $g$ are both convex, then their composition $f \circ g$ is not necessarily also convex. Consider for example convex functions $f(x) = x^2$ and $g(x) = x^2 - 1$. Then, the composition

$$(f \circ g)(x) = x^4 - 2x^2 + 1$$

satisfies $(f \circ g)(-1) = (f \circ g)(1) = 0$ and $(f \circ g)(0) = 1$, which is a clear violation of convexity.
Exercise 5. Prove Lemmata 1.27 and 1.28!

Solution: Proof of Lemma 1.27:

(i) ⇒ (ii): Let's fix some \( x \in \mathbb{R}^d \) and \( \lambda \geq 0 \). Vector \( y \) is a direction of recession of every nonempty sublevel set, in particular of the sublevel set \( f \leq f(x) \). Since \( x \) belongs to \( f \leq f(x) \), \( x + \lambda y \) also belongs to \( f \leq f(x) \), that is \( f(x + \lambda y) \leq f(x) \).

(ii) ⇒ (iii): A point \((0, f(0))\) belongs to \( \text{epi}(f) \), and it is enough to show that \((0 + \lambda y, f(0) + \lambda 0) = (\lambda y, f(0))\) is in \( \text{epi}(f) \) for all \( \lambda \geq 0 \). Since \( f(\lambda y) = f(0 + \lambda y) \leq f(0) \) for all \( \lambda \geq 0 \), every point \((\lambda y, f(0))\) is in \( \text{epi}(f) \).

(iii) ⇒ (i): We need to show that \( y \) is a direction of recession of the set \( f \leq f(0) \). Indeed, \((\lambda y, f(0)) = (0 + \lambda y, f(0) + \lambda 0)\) is in \( \text{epi}(f) \), hence \( f(0 + \lambda y) \leq f(0) \), and since \( 0 \) is in the set \( f \leq f(0) \), \( y \) is a direction of recession of \( f \leq f(0) \).

Proof of Lemma 1.28:

\( y \) is a direction of constancy of \( f \) (point (i)).

⇔ \( y \) and \(-y\) are directions of recession of \( f \).

⇔ (by Lemma 1.27) for all \( x \in \mathbb{R}^d \) and \( \lambda \geq 0 \), \( f(x + \lambda y) \leq f(x) \) and \( f(x) = f(x ± \lambda y = ± \lambda y) \leq f(x ± \lambda y) \).

⇔ for all \( x \in \mathbb{R}^d \) and \( \lambda \in \mathbb{R} \), \( f(x + \lambda y) = f(x) \) (point (ii)).

⇔ (by Lemma 1.27) \((y, 0)\) and \((-y, 0)\) are directions of recession of \( \text{epi}(f) \).

⇔ \((y, 0)\) is a direction of of constancy of \( \text{epi}(f) \) (point (iii)).

Exercise 6. Consider the function \( \ell \) defined in (1.10). Prove that \( \ell \) is convex!

Solution: It suffices to show that the function \(-\ln z_j(y)\) is convex for all \( j \), with \( z_j \) as in (1.9). Using Lemma 1.9 (i) and (ii), it then follows that \( \ell \) is convex. We compute

\[
-\ln z_j(y) = \ln (e^{y_0} + \cdots e^{y_9}) - y_j.
\]

The first summand is a log-sum-exp function and therefore convex (a proof goes via the Hessian [BV04, 3.1.5]). The second summand is a linear function and therefore also convex. Hence the sum is convex by Lemma 1.9 (i).

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Exercise 7. Consider the function $\ell$ defined in (1.10). Let us call an argument matrix $W$ a separator for $P$ if for all $x \in P$,

$$(Wx)_{d(x)} = \max_{j=0}^{9} (Wx)_j,$$

i.e. under (1.9), the correct digit has highest probability (possibly along with other digits). A separator is trivial if for all $x \in P$ and all $i, j \in \{0, \ldots, 9\}$,

$$(Wx)_i = (Wx)_j.$$

For example, whenever the rows of $W$ are pairwise identical, we obtain a trivial separator. But depending on the data, there may be other trivial separators. For example, if some pixel is black (gray value 0) in all images, arbitrarily changing the entries in the corresponding column of a trivial separator gives us another trivial separator. For a trivial separator $W$, (1.10) yields $\ell(W) = |P| \ln 10$.

Prove the following statement: $\ell$ has a global minimum if and only if all separators are trivial.

As a special case, consider the situation in which there exists a strong (and in particular nontrivial) separator: a matrix $W^*$ such that for all $x \in P$ and all $j \neq d(x)$,

$$(W^*x)_{d(x)} > (W^*x)_j,$$

i.e. the correct digit has unique highest probability. In this case, it is easy to see that $\ell(\lambda W^*) \to_{\lambda \to \infty} 0$, so we cannot have a global minimum, as $\inf_{W} \ell(W) = 0$ is not attainable.

Solution: At first we will prove that if $\ell$ has a global minimum, then all separators are trivial. We will show that if $W'$ is a nontrivial separator, then for every $W$, $\ell(W + \lambda W') < \ell(W)$ for all $\lambda > 0$. So if there exists a nontrivial separator, we can always decrease the value of $\ell$ and hence $\ell$ cannot have a global minimum.

Fix some matrix $W$, some number $\lambda > 0$ and some nontrivial separator $W'$. By definition of a nontrivial separator, there exists some $x_0$ and a digit $i_0$ such that $(W'x_0)_{i_0} < (W'x_0)_{d(x_0)}$ and $(W'x)_j \leq (W'x)_{d(x)}$ for all $x \in P$. 

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and digits $j$. We get:

$$\ell(W + \lambda W') =$$

$$= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} e^{\lambda(W'x)_j} \right) - (Wx)_d(x) - \lambda(W'x)_d(x) \right)$$

$$= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} e^{\lambda(W'x)_j} + e^{(Wx)_{i_{0}}} e^{\lambda(W'x)_{i_{0}}} \right) - (Wx)_d(x) - \lambda(W'x)_d(x) \right)$$

$$\leq \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} e^{\lambda(W'x)_d(x)} + e^{(Wx)_{i_{0}}} e^{\lambda(W'x)_{d(x)}} \right) - (Wx)_d(x) - \lambda(W'x)_d(x) \right)$$

$$= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} \right) + \ln(e^{\lambda(W'x)_d(x)}) - (Wx)_d(x) - \lambda(W'x)_d(x) \right)$$

$$= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} \right) - (Wx)_d(x) \right) = \ell(W)$$

Now let’s prove that if all separators are trivial, then $\ell$ has a global minimum. We will show that the recession cone of $\ell$ is the set of all separators and that the lineality space of $\ell$ contains all trivial separators. Hence if all separators are trivial, the recession cone of $\ell$ is equal to the lineality space of $\ell$ and by Theorem 1.33 $\ell$ has a global minimum.

At first let’s show that the lineality space of $\ell$ contains all trivial separators. Fix some matrix $W$, some real number $\lambda$ and some trivial separator $W'$. By definition of a trivial separator, for all $x \in P$ and digits $j$,
\((W'x)_j = (W'x)_{d(x)}\). We get:
\[
\ell(W + \lambda W') =
\sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} e^{\lambda(W'x)_j} \right) - (Wx)_{d(x)} - \lambda(W'x)_{d(x)} \right)
\]
\[
= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} e^{\lambda(W'x)_{d(x)}} \right) - (Wx)_{d(x)} - \lambda(W'x)_{d(x)} \right)
\]
\[
= \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{(Wx)_j} - (Wx)_{d(x)} \right) = \ell(W) \right)
\]

By Lemma 1.28 \(W'\) belongs to the lineality space of \(\ell\).

It remains to show that the recession cone of \(\ell\) is the set of all separators. We have already shown that if \(W'\) is a separator, then \(\ell(W + \lambda W') \leq \ell(W)\) for all \(\lambda > 0\) (for nontrivial separators this inequality is strict and for trivial separators this inequality is actually an equality). Hence by Lemma 1.27 all separators are directions of recession.

Let’s show that any direction of recession \(V\) of \(\ell\) is a separator. Assume that it is not true: \(\ell(0 + \lambda V) \leq \ell(0)\) for all \(\lambda > 0\), and there exists some \(x_0 \in P\) and a digit \(i_0\) such that \((Vx_0)_{i_0} > (Vx_0)_{d(x_0)} + \varepsilon\) for some \(\varepsilon > 0\). For all \(\lambda > 0\) we get:
\[
\ell(0 + \lambda V) = \sum_{x \in P} \left( \ln \left( \sum_{j=0}^{9} e^{\lambda(Vx)_j} \right) - \lambda(Vx)_{d(x)} \right)
\]
\[
\geq \ln \left( \sum_{j=0}^{9} e^{\lambda(Vx_0)_j} \right) - \lambda(Vx_0)_{d(x_0)}
\]
\[
\geq \ln e^{\lambda(Vx_0)_{i_0}} - \lambda(Vx_0)_{d(x_0)} > \lambda \varepsilon
\]

The first inequality follows from the fact that the value
\[
\ln \left( \sum_{j=0}^{9} e^{(Vx)_j} \right) - (Vx)_{d(x)} = \ln \frac{\sum_{j=0}^{9} e^{(Vx)_j}}{e^{(Vx)_{d(x)}}}
\]
is positive for all \(V\) and \(x\) since the argument of \(\ln\) is greater than 1.

For large enough \(\lambda\), \(\ell(0 + \lambda V) > \lambda \varepsilon > \ell(0)\). This contradiction finishes the proof.
Exercise 8. Prove that the function $f(x) = \|x\|_1 = \sum_{i=1}^{d} |x_i|$ ($\ell_1$-norm) is convex!

Solution: It suffices to prove that $f_i(x) = |x_i|$ is convex and then use Lemma 1.9. Equivalently, that $f(x) = |x|$ is convex. For $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we compute

$$f(\lambda x + (1 - \lambda) y) = \lambda |x| + (1 - \lambda) |y| \leq |\lambda x| + |(1 - \lambda)y| \quad (\text{triangle inequality})$$

$$= |\lambda||x| + |(1 - \lambda)||y|$$

$$= \lambda |x| + (1 - \lambda) |y|$$

$$= \lambda f(x) + (1 - \lambda) f(y).$$

Exercise 9. A seminorm is a function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying the following two properties for all $x, y \in \mathbb{R}^d$ and all $\lambda \in \mathbb{R}$.

(i) $f(\lambda x) = |\lambda| f(x)$,

(ii) $f(x + y) \leq f(x) + f(y)$ (triangle inequality).

Prove that every seminorm is convex!

Solution: This just generalizes the previous exercise and shows what is actually going on. For $\lambda \in [0, 1]$ we get

$$f(\lambda x + (1 - \lambda) y) \leq f(\lambda x) + f((1 - \lambda) y) \quad (\text{triangle inequality})$$

$$= |\lambda| f(x) + |(1 - \lambda)| f(y)$$

$$= \lambda f(x) + (1 - \lambda) f(y).$$

Exercise 10. Suppose that we have centered observations $(x_i, y_i)$ such that $\sum_{i=1}^{n} x_i = 0, \sum_{i=1}^{n} y_i = 0$. Let $w^*$ be the global minimum of the least squares objective

$$f(w^*) = \sum_{i=1}^{n} (w_0 + \sum_{j=1}^{d} w_j x_{ij} - y_i)^2.$$

Prove that $w^*_0 = 0$.

Solution: We compute

$$\frac{\partial f(w)}{\partial w_0} = 2 \sum_{i=1}^{n} (w_0 + \sum_{j=1}^{d} w_j x_{ij} - y_i) = 2 \sum_{i=1}^{n} w_0 = 2nw_0.$$
since the observations are centered. Also, by Lemma 1.13

\[ 0 = \frac{\partial f(w)}{\partial w_0} \bigg|_{w = w^*} = 2nw^*_0. \]