Topological Methods in Combinatorics and Geometry  FS 08

Problem Set 11

Course webpage: http://www.ti.inf.ethz.ch/ew/courses/Top08/

Due date: May 29, 2008

Exercise 1. Find a simplicial complex $K$ such that $||K^2\Delta||$ is homeomorphic to $S^1 \times [0, 1]$.

Exercise 2 (n-dimensional space that does not embed into $\mathbb{R}^{2n}$). Let $A$ be a three point discrete space and $B$ be a singleton space. The 3-star is the space $Y = A * B$ drawn in the picture (we consider $A$ and $B$ as subspaces of $Y$).

Next, we define the space:

$$N = \{(y_1, y_2, \ldots, y_{n+1}) \in Y^{n+1} \mid y_i \in A \text{ for at least one } i \in [n+1]\}.$$  

The topology on $N$ is the inherited topology of $N$ as a subspace of the Cartesian product $Y^{n+1}$.

(a) Find a map $f: B^2 \rightarrow Y$ such that $f(x) \neq f(-x)$ for every $x \in S^1 = \partial B^2$.

(b) Prove that $Y^{n+1}$ does not embed into $\mathbb{R}^{2n+1}$.

(c) Prove that if $N$ embeds into $\mathbb{R}^k$ then $Y$ embeds into $\mathbb{R}^{k+1}$, and thus conclude that $N$ does not embed into $\mathbb{R}^{2n}$.

Exercise 3. Let $V_{n,2} = \{(v_1, v_2) \in (S^{n-1})^2 \mid \langle v_1, v_2 \rangle\} \subset \mathbb{R}^{2n}$ be the Stiefel manifold of pairs of unit orthogonal vectors, $n \geq 1$. Let $\nu$ be the $\mathbb{Z}_2$-action given by $(v_1, v_2) \mapsto (-v_1, -v_2)$.

(a) Show that $\text{ind}_{\mathbb{Z}_2}(V_{2,n}) \leq n - 1$.

(b) Let $n$ be even. Exhibit a $\mathbb{Z}_2$-map $S^{n-1} \rightarrow V_{n,2}$, thereby proving that $\text{ind}_{\mathbb{Z}_2}(V_{n,2}) = n - 1$.

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1The space $N$ can be seen as a geometric realization of some $n$-dimensional simplicial complex, but you are not supposed to prove it.

2If we consider $Y$ as subspace of $\mathbb{R}^2$, then we can see $Y^{n+1}$ as a subspace of $\mathbb{R}^{2(n+1)}$. 