The Hales-Jewett Theorem

A.W. HALES & R.I. JEWETT

1963

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Importance of HJT

- The Hales-Jewett theorem is presently one of the most useful techniques in *Ramsey theory*

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- Without this result, Ramsey theory would more properly be called Ramseyan theorems

Outline for the lecture

1. The theorem and its consequences
    1.1 Van der Waerden's Theorem
    1.2 Gallai-Witt's Theorem

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2. Historical Comments
3. Proof of HJT

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- Let $* \notin A$ be a new symbol
- *Words* are strings over A without $*$
- Strings containing at least one $*$ are called *roots*
- For a root $\tau \in (A \cup \{*\})^n$ and a symbol $a \in A$, we write

$$\tau(a) \in A^n$$

  for the word obtained from τ by replacing each $*$ by a

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- Some words obtained from these roots:

$\tau_1(0) = 0102 \quad \tau_1(1) = 0112 \quad \tau_2(1) = 11$
or $\tau_3(2) = 012221$.

Further definition

Let τ be a root. A *combinatorial line* rooted in τ is the set of t words

$$L_\tau = \{\tau(0), \tau(1), \ldots, \tau(t-1)\}.$$

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$$L_\tau = \left\{ \begin{array}{cccccc} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 1 & 3 & 2 & 3 & 1 \\ 0 & 1 & 4 & 2 & 4 & 1 \end{array} \right\} \qquad |L_\tau| = 5.$$

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$$(|A| + 1)^n - |A|^n$$

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\end{aligned}$$

■

Why combinatorial line?

Let $A := \{0, 1, 2\}$, n=3 and $\tau = (*, 2, *)$. Then

$$L_\tau = \left\{ \begin{array}{ccc} 0 & 2 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{array} \right\}$$

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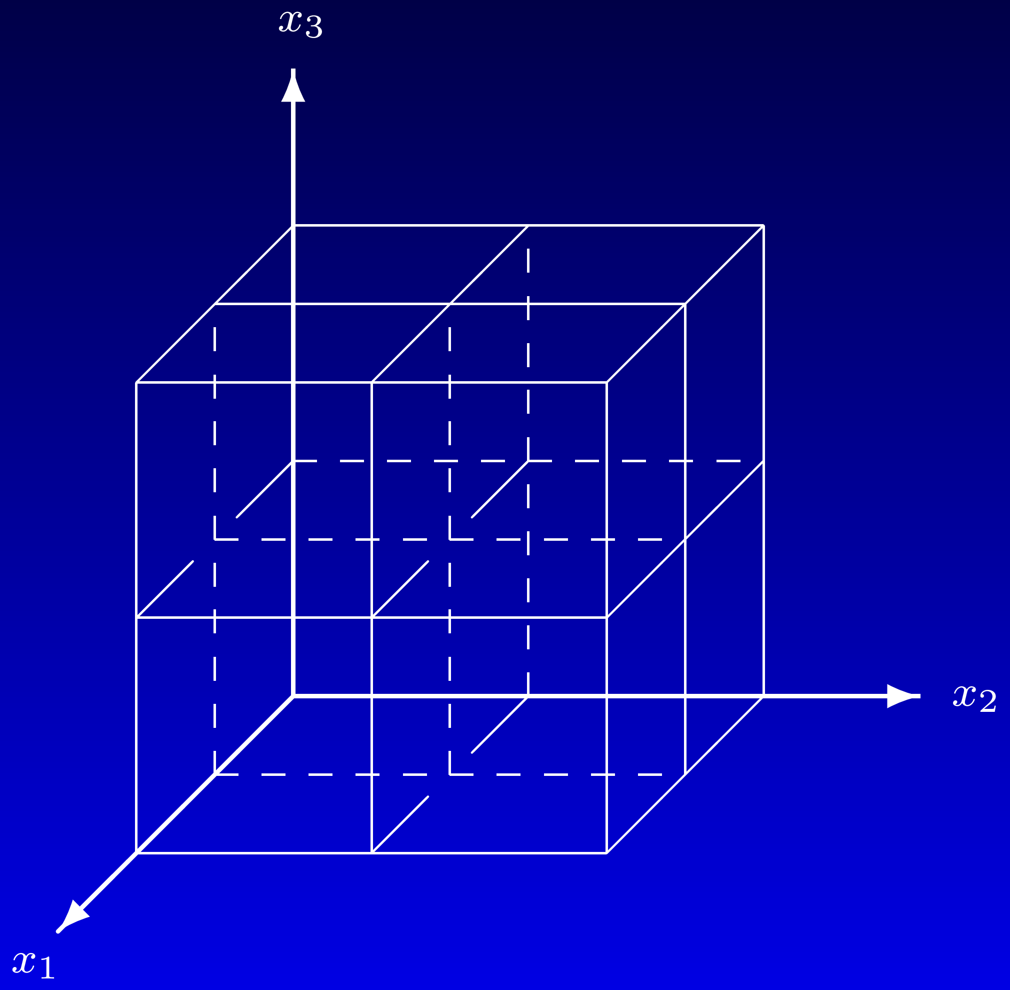

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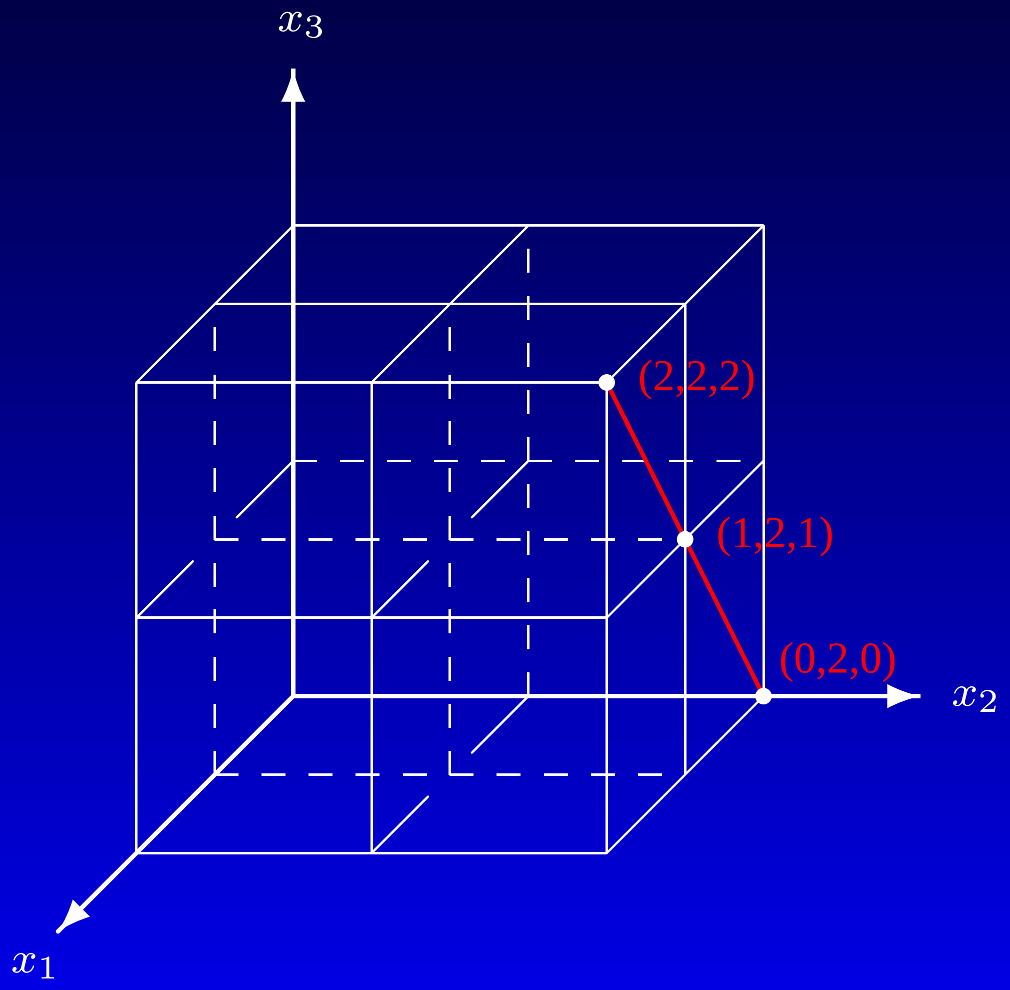

Theorem (HJT, 1963)

Given $r > 0$ distinct colors and a finite alphabet A of $t = |A|$ symbols.

There is a dimension $n = HJ(r, t) \in \mathbb{N}$ such that for every r-coloring of the cube A^n there exists a monochromatic combinatorial line.

Examples

- Take $r = \#\{\text{colors}\} = 1$.
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- Let $A = \{0, 1\}$ (A^n then corresponds to Q^n).

  $$r = 2: \quad \Rightarrow \quad HJ(2, 2) =$$

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  $$r = 3\text{:} \quad \Rightarrow \quad HJ(3,2) = \ 3$$

Van der Waerden (1927)

Theorem: Given $r > 0$ distinct colors and $t \in \mathbb{N}$.

There is a $N = W(r,t) \in \mathbb{N}$ such that for every r-coloring of the set $\{1, \ldots, N\}$ there exists at least one monochromatic arithmetic progression of t terms:

$$\{1, \ldots, a, \ldots, a + d, \ldots, a + 2d, \ldots, a + (t-1)d, \ldots, N\}$$

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Set $A := \{0, 1, \ldots, t-1\}$ and define for $x = (x_1, \ldots, x_n) \in A^n$ the mapping

$$\begin{aligned} f : A^n &\rightarrow \{1, 2, \ldots, N\} \\ x &\mapsto x_1 + x_2 + \cdots + x_n + 1. \end{aligned}$$

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$$\begin{aligned} f : A^n &\to \{1, 2, \ldots, N\} \\ x &\mapsto x_1 + x_2 + \cdots + x_n + 1. \end{aligned}$$

Thus, f induces a coloring of A^n

color of $x \in A^n :=$ color of number $f(x)$.

Proof (cont.)

Every combinatorial line

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$$\begin{aligned} a &:= f(\tau(0)) + 1 \\ d &:= \#\{*\text{'s in } \tau\} \end{aligned}$$

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$$\begin{aligned} a &:= f(\tau(0)) + 1 \\ d &:= \#\{*\text{'s in } \tau\} \end{aligned}$$

By HJT there is a monochromatic line in A^n, which corresponds to a monochromatic arithmetic progression of length t. ∎

Gallai-Witt (1943,1951)

A set of vectors $U \subseteq \mathbb{Z}^m$ is a *homothetic copy* of $V \subseteq \mathbb{Z}^m$ if there is a vector $u \in \mathbb{Z}^m$ and a constant $\lambda \in \mathbb{N}$, such that

$$U = u + \lambda V := \{u + \lambda v | v \in V\}$$

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Theorem: Given $r > 0$ distinct colors and let the vectors of $\mathbb{Z}^m$ be r-colored.

Then every finite subset $V \subset \mathbb{Z}^m$ has a homothetic copy which is monochromatic.

Proof (using HJT)

Let $r := \#\{\text{colors}\}$ and $V := \{v_0, \ldots, v_{t-1}\} = A$. Set $n = HJ(r, t)$ and consider

$$A^n \ni x = (x_1, \ldots, x_n) \qquad x_i \in \{v_0, \ldots, v_{t-1}\}.$$

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Define $f : A^n \to \mathbb{Z}^m$ by $f(x) = x_1 + \cdots + x_n$.
This induces a r-coloring of A^n.

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By HJT there is a monochromatic combinatorial line

$$L_\tau = \{\tau(v_0), \ldots, \tau(v_{t-1})\} \subseteq A^n.$$

Proof (cont.)

Let $I = \{i | \tau_i \neq *\}$, and

$$\lambda := \#\{*\text{'s in } \tau\} = n - |I| > 0.$$

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$$f(L_\tau) = \left\{ u + \lambda v_j \;\middle|\; u = \sum_{i \in I} v_i \; ; \; j = 0, \ldots, t-1 \right\}$$

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This is a homothetic copy of $V = \{v_0, \ldots, v_{t-1}\}$. ■

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- Fürstenberg and Katznelson proved Szemerédi's theorem using ergodic theory in 1979.
- Gowers gave a new proof (1998), with a better bound on $W(r, t)$, for the case $t = 4$ (mentioned in his Fields Medal citation).

Known $W(r, t)$

- The following table shows all exactly known van der Waerden numbers (the nontrivials marked red)

r/t	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$r = 1$	1	2	3	4	5
$r = 2$	1	3	9	35	178
$r = 3$	1	4	27		
$r = 4$	1	5	76		

Eric W. Weisstein et al. ”van der Waerden Number.” From MathWorld–A Wolfram Web Resource.

`http://mathworld.wolfram.com/vanderWaerdenNumber.html`

Some bounds

It is known that

$$W(r,3) \leq e^{r^{c_1}}$$

and

$$W(r,4) \leq e^{e^{e^{r^{c_2}}}}$$

for some constants c_1, c_2.

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  A function that can be implemented using only `for`-loops (which have a fixed iteration limit) is called primitive recursive.
- We will follow the compact version of Shelah's proof from A. Nilli (1990) (c/o Noga Alon).

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Set $n := HJ(r, t - 1)$ and define a sequence $N_1 < \ldots < N_n$

$$N_1 := r^{t^n} \qquad N_i := r^{t^n + \sum_{j=1}^{i-1} N_j} \quad i \geq 2$$

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Define as a new Dimension

$$N := N_1 + \cdots + N_n.$$

We want to prove that for every coloring χ of the N-cube A^N

$$\chi : A^N \to \{1, \ldots, r\}$$

there is at least one monochromatic combinatorial line, that is, $HJ(r,t) \leq N$.

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How large is this N?

We have $n = HJ(r, t-1)$. Estimate N

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Example: Take $r = 2$ and $t = 3$. $n = HJ(2,2) = 2$. Then

$$N > 2^{2^{3^2}} = 2^{512}$$

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Example: $a, b \in A^n$ differ in the i-th coordinate

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For $a = a_1 \dots a_n \in A^n$ and a concatenation of n roots

$$\tau = \tau_1 \dots \tau_n \qquad |\tau_i| = N_i \quad \forall i = 1, \dots, n$$

denote

$$\tau(a) = \tau_1(a_1) \dots \tau_n(a_n) \qquad |\tau(a)| = N.$$

Claim

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$$\chi(\tau(a)) = \chi(\tau(b))$$

for any two neighbors $a, b \in A^n$.

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How does this imply the theorem?

Take such a τ from the claim and define a new coloring χ' for the n-cube $(A - \{0\})^n$

$$\chi'(a) := \chi(\tau(a)) \qquad \chi : A^N \rightarrow \{1, \ldots, r\}.$$

Since $|A - \{0\}| = t - 1$ and $n = HJ(r, t-1)$ by induction assumption there is a root

$$\nu = \nu_1 \dots \nu_n \in \big((A - \{0\}) \cup \{*\}\big)^n$$

such that the combinatorial line

$$L_\nu = \{\nu(1), \nu(2), \dots, \nu(t-1)\}$$

is monochromatic with respect to χ'.

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Consider the root

$$\tau(\nu) = \tau_1(\nu_1) \ldots \tau_n(\nu_n) \qquad |\tau(\nu)| = N.$$

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We will show that the line

$$L_{\tau(\nu)} = \big\{\tau(\nu(0)), \tau(\nu(1)), \ldots, \tau(\nu(t-1))\big\}$$

is monochromatic with respect to χ.

Since L_ν is monochromatic w.r.t. χ'

$$\chi'(\nu(1)) = \ldots = \chi'(\nu(t-1)).$$

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If $\#\{*\text{'s in } \nu\} = 1$, then $\tau(\nu(0))$ is neighbor of $\tau(\nu(1))$. By the claim

$$\chi\big(\tau(\nu(0))\big) = \chi\big(\tau(\nu(1))\big).$$

If $\#\{*\text{'s in } \nu\} > 1$, then we still can reach $\tau(\nu(0))$ from $\tau(\nu(1))$ by passing through a sequence of neighbors

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Again by the claim above

$$\chi\big(\tau(\nu(0))\big) = \chi\big(\tau(\nu(1))\big).$$

Thus, $L_{\tau(\nu)}$ is monochromatic. ∎

Proof (of claim)

We prove the existence of roots τ_i by backward induction on i.

Suppose we already have defined $\tau_{i+1}, \ldots, \tau_n$.

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For $k = 0, \ldots, N_i$ let W_k be the word

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For each k define the r-coloring χ_k of all words in $A^{M_{i-1}+n-i}$ as

$$\chi_k\big(x_1 \ldots x_{M_{i-1}} y_{i+1} \ldots y_n\big) :=$$
$$:= \quad \chi\big(x_1 \ldots x_{M_{i-1}}\, W_k\, \tau_{i+1}(y_{i+1}) \ldots \tau_n(y_n)\big).$$

Proof (cont.)

We have $N_i + 1$ colorings $\chi_0, \ldots, \chi_{N_i}$.
The total number of such r-colorings is

$$r^{\#\{\text{of words}\}} = r^{t^{M_{i-1}+n-i}} \leq r^{t^{M_{i-1}+n}} = N_i.$$

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We have to check that $\tau = \tau_1 \ldots \tau_n$ satisfies the assertion of the claim.

Proof (cont.)

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$$\tau_i(0) = W_k \quad \text{and} \quad \tau_i(1) = W_s.$$

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Take $a, b \in A^n$ neighbors differing in the i-th coordinate

$$\begin{aligned} a &= a_1 \ldots a_{i-1}\, 0\, a_{i+1} \ldots a_n \\ b &= a_1 \ldots a_{i-1}\, 1\, a_{i+1} \ldots a_n \end{aligned}$$

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then

$$\begin{aligned} \tau(a) &= \tau_1(a_1) \ldots \tau_{i-1}(a_{i-1})\, \tau_i(0)\, \tau_{i+1}(a_{i+1}) \ldots \tau_n(a_n) \\ \tau(b) &= \tau_1(a_1) \ldots \tau_{i-1}(a_{i-1})\, \tau_i(1)\, \tau_{i+1}(a_{i+1}) \ldots \tau_n(a_n) \end{aligned}$$

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and since $\chi_s = \chi_k$

Proof (cont.)

$$\chi(\tau(a)) = \chi\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\ \tau_i(0)\ \tau_{i+1}(a_{i+1})\ldots\tau_n(a_n)\big)$$

Proof (cont.)

$$\begin{aligned}\chi(\tau(a)) &= \chi\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\;\tau_i(0)\;\tau_{i+1}(a_{i+1})\ldots\tau_n(a_n)\big)\\ &= \chi\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\;W_k\;\tau_{i+1}(a_{i+1})\ldots\tau_n(a_n)\big)\end{aligned}$$

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Proof (cont.)

$$
\begin{aligned}
\chi(\tau(a)) &= \chi\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\ \tau_i(0)\ \tau_{i+1}(a_{i+1})\ldots\tau_n(a_n)\big) \\
&= \chi\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\ W_k\ \tau_{i+1}(a_{i+1})\ldots\tau_n(a_n)\big) \\
&= \chi_k\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\ a_{i+1}\ldots a_n\big) \\
&= \chi_s\big(\tau_1(a_1)\ldots\tau_{i-1}(a_{i-1})\ a_{i+1}\ldots a_n\big)
\end{aligned}
$$

Proof (cont.)

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&= \chi(\tau(b))
\end{aligned}$$

This completes the proof of the claim. ■

Combinatorial m-space

Let τ be a root in $(A \cup \{*_1, \ldots, *_m\})^n$, where $*_1$,. . .,$*_m \notin A$ are distinct symbols. We require that each of these appears at least once in τ. Thus, we have m mutually disjoint sets of moving coordinates.

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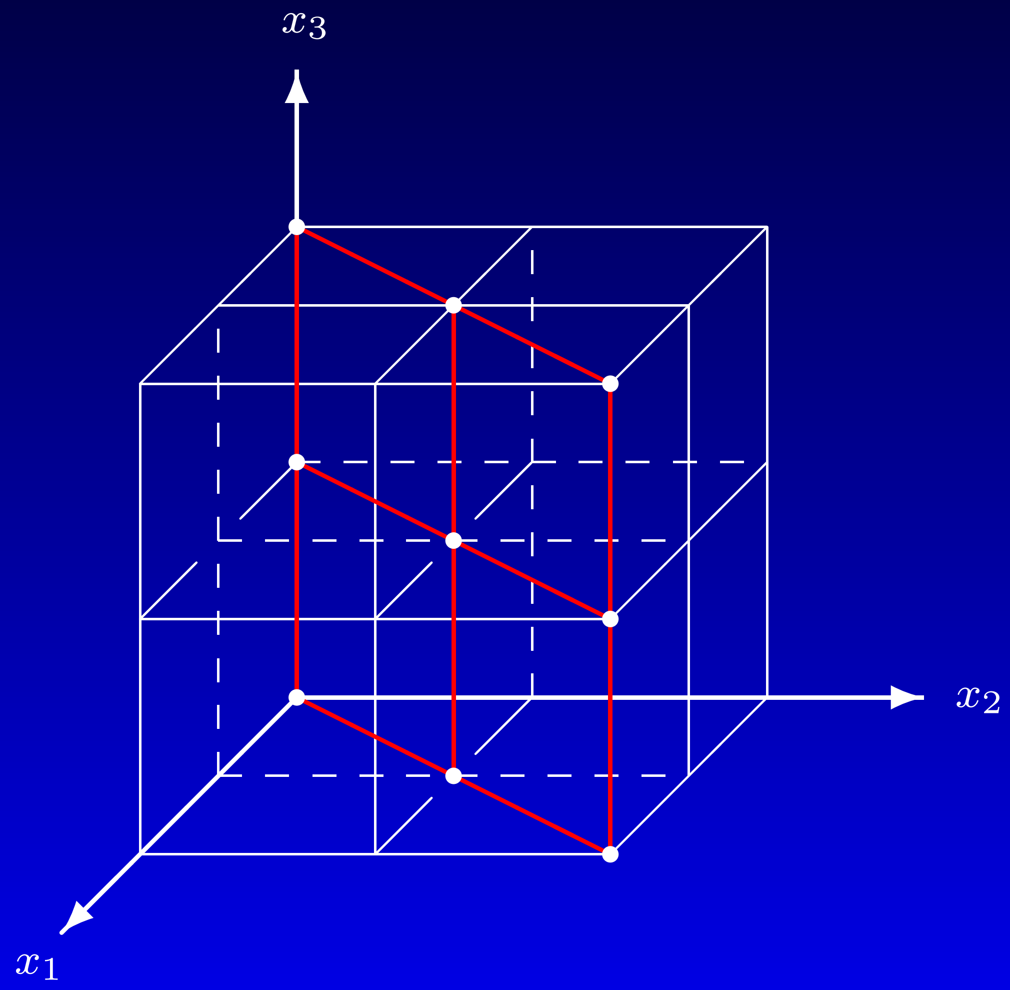
Remark: A combinatorial line is a combinatorial 1-space.

Example

A combinatorial 2-space in $A^3 = \{0, 1, 2\}^3$.



Here $\tau = (*_1, *_1, *_2)$.

Theorem (generalized)

Given $r > 0$ distinct colors, a dimension $m \in \mathbb{N}$ and a finite alphabet A of $t = |A|$ symbols.

There is a dimension $n = HJ(m, r, t) \in \mathbb{N}$ such that for every r-coloring of the cube A^n there exists a monochromatic combinatorial m-space.

End