Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing — Exercises KW38 — HS16

The following exercises will be solved in the first exercise class on September 21, 2016. Since all exercises are in-class, there is no due date.

Exercise 1: Dependency (in-class)

We consider two fair dice (i.e. every number appears with probability $\frac{1}{6}$). Let X_1 and X_2 denote the random variable for the outcome of the first die and the second die, respectively.

(a) Compute $\Pr[X_1 + X_2 = 8]$.

(b) Compute $\Pr[X_1 + X_2 \geq 6 \mid X_1 \leq 2]$.

(c) Determine for each of the following pairs $\mathcal{E}_1, \mathcal{E}_2$ of events whether they are dependent:

(i)
- $\mathcal{E}_1$: X_1 is even.
- $\mathcal{E}_2$: $X_1 + X_2$ is odd.

(ii)
- $\mathcal{E}_1$: X_1 is even.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 8$.

(iii)
- $\mathcal{E}_1$: $X_1 = X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \geq 10$.

(iv)
- $\mathcal{E}_1$: $X_1 \geq X_2$.
- $\mathcal{E}_2$: $X_1 + X_2 \leq 3$.

Exercise 2: Geometric Distributions (in-class)

You are given a randomized algorithm which has a probability $p \in (0, 1)$ of computing the correct answer. With probability $1 - p$ the algorithm will output "FAILURE".

(a) How many times do we need to run the algorithm in expectation until we learn the correct answer?

(b) Let $p = 1/2$. What is the probability that we have to run the algorithm an even number of times until we learn the correct answer?

(c) Same question for general $p \in (0, 1)$.

Exercise 3: Expected running time (in-class)

Let A be a randomized algorithm and let X be the random variable for the running time of A. (We assume for this exercise that X is independent of the input.) The following table gives the probability distribution for X.

k	1	2	3
$\Pr[X = k]$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(a) Compute $\mathbf{E}[X]$, $\mathbf{E}\left[X^2\right]$, and $\mathbf{E}[X]^2$.

(b) We run A twice in a row and write X_1, X_2 for the running time of the first and second call to A, respectively.

  (i) What is the expected overall running time?

  (ii) Compute $\mathbf{E}[X_1 \cdot X_2]$.

  (iii) Compute the probability that the overall running time is at most 4.

(c) Let B be an algorithm which repeatedly calls A as a subroutine. Suppose that the number of subroutine calls is described by a random variable N.

  Prove or disprove: The expected overall running time equals $\mathbf{E}[X] \cdot \mathbf{E}[N]$.

Exercise 4: Random Walks (in-class)

An apple is located at vertex A of a pentagon $ABCDE$, and a worm is located at C. Every day the worm crawls with equal probability to one of the two adjacent vertices. When it reaches vertex A, it stops to dine.

(a) What is the expected number of days until dinner?

(b) What does Markov's Inequality say about the probability that the number of days is 100 or more?

Exercise 5: Independence of Three Events (in-class)

Give an example of a probability space with three events A, B, C such that both of the following hold:

- The events are pairwise independent.
- The events are *not* mutually independent.

Exercise 6: Conditional Probability (in-class)

(a) Suppose you have an egg and a bottle of milk in your fridge. Each of them is spoiled with probability exactly 0.5, independently of the other. Assume that you inspect the milk and find out that it is spoiled. What is now the probability that the egg is spoiled as well?

(b) Suppose a new family moves into your neighborhood. All you know about them is that they have two children. At some point you get the information that one of the two children is called Markus. What is the probability that the other child is a boy as well?

For simplicity, we assume that each child is a boy with probability exactly 0.5, independently of its siblings.

Exercise 7: Paradoxes (in-class)

(a) In the famous *Monty Hall Problem*, you are on a TV show, facing three closed doors on the stage. You know that behind two doors there is a goat, and behind one there is a beautiful car. The rules are as follows: In step 1, you point at some door. In step 2, the show master opens one of the remaining doors, but he is not allowed to open the door with the car behind it. In step 3 you can point to one of the closed doors, and get as a prize whatever is behind it.

Now assume that you have picked door 1 in step 1, the car is behind a door uniformly at random, and the show master picks uniformly at random if he has the choice. You see the show master has opened door 2. Should you now pick door 1 or 3?

(b) Now assume the show master does not pick uniformly at random. If the car is behind door 1, he picks door 2 with probability p and door 3 with probability $1-p$. What is your optimal strategy now if the show master opens door 2? What if he opens door 3? What is the overall probability to get the car?

(c) Harry and Hermione are in the laboratory of an evil wizard. On the desk, there are three cups with a potion in them. Harry and Hermione know that two of them are poisonous, but one gives unlimited power. They also know that the evil wizard will return soon and kill them if neither of them drinks the "good" potion. Harry randomly chooses a cup and lifts it to his lips (not yet drinking). Hermione chooses randomly one of the other two, drinks it — and dies. What should Harry do? Should he drink from the cup he is holding, or should he switch to the third cup, or does it not matter? Note that running away is not an option.