ETH
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Algorithms, Probability, and Computing — Exercises KW45 — HS16

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A/B: Wed 13–15 CAB G 56

  Group C: Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

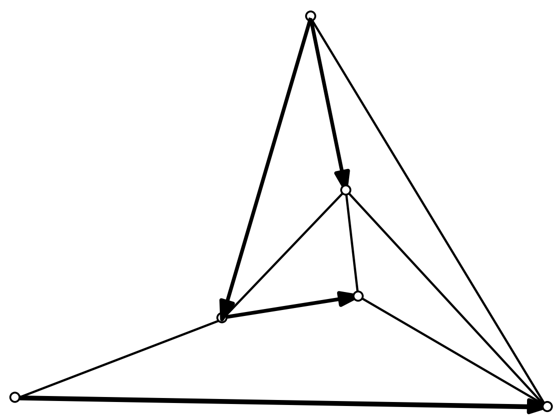
The following exercises will be discussed in the exercise class on November 9, 2016. Please hand in your solutions not later than November 8.

Exercise 1: Randomized Algebraic Algorithms (in-class)

(a) For $n \in \mathbf{N}$, let $A \in \mathbf{R}^{n\times n}$ be a non-zero matrix (i.e. not all entries are 0) and let x be a vector u.a.r. from $\{-1, 0, +1\}^n$. Show that the probability that the vector Ax is non-zero is at least $2/3$.

(b) Let G be a graph with no cycle of even length. Show that there is at most one perfect matching in G.

(c) Complete the following partial orientation (four edges are already oriented) to a Pfaffian orientation.



Exercise 2: Checking Matrix Multiplication

(a) Let K be any field. Show that one iteration of the checking algorithm for matrices $A, B, C \in K^{n \times n}$ detects an error in the supposed product matrix C with probability at least $\frac{1}{2}$.

(b) Suppose $K = GF(2)$ (i. e., all calculations with matrices are done modulo 2), and suppose the matrix C is wrong in exactly one row. Show that in one iteration the success probability of detecting an error in the supposed product matrix C is exactly $\frac{1}{2}$.

Exercise 3: The Schwartz-Zippel Theorem is Tight

Given a finite set S of rational numbers and positive integers d and n, $d \leq |S|$, find a polynomial $p(x_1, x_2, \ldots, x_n)$ of degree d for which the Schwartz–Zippel theorem is tight. That is, the number of n-tuples $(r_1, \ldots, r_n) \in S^n$ with $p(r_1, \ldots, r_n) = 0$ is $d|S|^{n-1}$.

Exercise 4: The Permanent and the Determinant

Let A be an $n \times n$ matrix with 0/1-entries. For $1 \leq i, j \leq n$ let $\epsilon_{i,j}$ be independent random variables, $\epsilon_{i,j} \in_{u.a.r.} \{-1, +1\}$. Let B be the random matrix with $b_{i,j} = \epsilon_{i,j} \cdot a_{i,j}$. In other words, to get B from A we randomly assign signs to the entries of A.

(a) Show that $\mathbf{E}[\det B] = 0$.

(b) Show that $\mathbf{E}\left[(\det B)^2\right] = \mathrm{per}(A)$.

Exercise 5: Existence vs. Explicit Construction of Matchings

Suppose that we have an algorithm for testing the existence of a perfect matching in a given graph, with running time at most $T(n)$ for any n-vertex graph.

(a) Explain how repeated calls to the algorithm can be used to find a perfect matching if one exists. Estimate the running time of the resulting algorithm.

(b) How can the algorithm be used for finding a maximum matching in a given graph?