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Algorithms, Probability, and Computing In-Class Exercises KW42 HS16

The following exercises will be discussed in the exercise class on October 19, 2016. Since we expect you to be working on the special assignment, all exercises are in-class.

In-Class Exercise 1: Number of Cells in Arrangements

We prove lemma 2.5:

Recall why an arrangement of n lines has at most $\binom{n}{2}$ vertices and at most n^2 edges. Show that it has at most $\binom{n+1}{2} + 1$ cells, and show that of all these bounds are attained if no three lines intersect in a common point and no two lines are parallel.

In-Class Exercise 2: Locating a Point in a Line Arrangement

Recall the algorithm for counting the points below a query line: We interpreted the question in the dual setting, so we were looking for the number of lines above a query point. Prove the space bound in lemma 2.6, which was left as an exercise:

$$\sum_{v \text{ inner node}} |\bar{S}_v| \leq 2n^2$$

where, as you may recall,

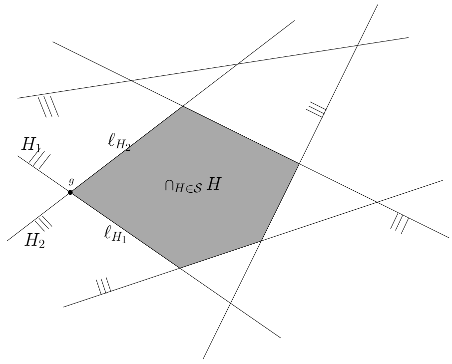
- n is the number of lines,
- v ranges over the nodes of the data structure used in the algorithm, which has one node for every level of the line arrangement;
- S_v is the set of x-coordinates of the corresponding level;
- $\bar{S}_v$ is the 'enhanced' set: If a node v has no child which is an inner node, $\bar{S}_v = S_v$. Otherwise, $\bar{S}_v$ is obtained from S_v by adding every other value from each of the sets $\bar{S}_u$, u a non-leaf child of v.

In-Class Exercise 3: Nearest Neighbor Changes

We are given a set P of n points in $\mathbf{R}^2$ and a point q which has distinct distances to all points in P. We add the points of P in random order (starting with the empty set), and observe the nearest neighbor of q in the set of points inserted so far. What is the expected number of distinct nearest neighbors that appear during the process?

In-Class Exercise 4: Linear Separability in Linear Time

Suppose we are given a set $\mathcal{S}$ of n closed halfspaces in the plane. For each $H \in \mathcal{S}$, let $\ell_H \subset H$ denote its boundary line. We assume that the halfspaces are in general position such that no two boundary lines are parallel and no three boundary lines meet in a single point. Consider the input to be given in the form of linear inequalities, say.



In this task we are interested in a randomized algorithm to decide whether the intersection of the given halfspaces is non-empty, that is whether $R(\mathcal{S}) = \emptyset$ for $R(\mathcal{S}) := \bigcap_{H \in \mathcal{S}} H$, or not. If $\mathcal{S}$ has a non-empty intersection, we would also be interested in a *certificate point*, that is in a point $x \in \bigcap_{H \in \mathcal{S}} H$ to demonstrate non-emptiness. To make your calculations simpler, we want to make certificate points unique. To this end, we assume $|\mathcal{S}| \geq 2$ and fix, arbitrarily, two halfspaces $H_1, H_2, \in \mathcal{S}$. The region $R(\mathcal{S})$ is obviously contained in a wedge formed by the lines ℓ_{H_1} and ℓ_{H_2} (see figure). Before starting any algorithm, you may assume that the input is rotated[1] first in such a way that this wedge opens to the right and the intersection point $g \in \ell_{H_1} \cap \ell_{H_2}$ acts as a guard that no point in $R(\mathcal{S})$ can have a smaller x-coordinate than g (see figure). We then define for any $\mathcal{S}' \subseteq \mathcal{S}$ with $H_1, H_2 \in \mathcal{S}'$ the unique certificate point $c(\mathcal{S}')$ as the point in $R(\mathcal{S}')$ that has the smallest x-coordinate. You may assume that H_1 and H_2 are fixed before and known to all your algorithms below.

Following are your tasks:

(a) Let $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above) and let $H \in \mathcal{S} \setminus \{H_1, H_2\}$ be an arbitrary one of the halfspaces. Prove: if $R(\mathcal{S}) \neq \emptyset$, then either $c(\mathcal{S}) = c(\mathcal{S} \setminus \{H\})$ or $c(\mathcal{S}) \in \ell_H$.

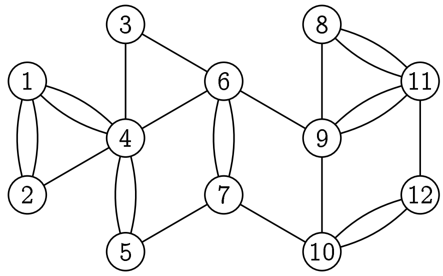
(b) Let $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above) and let $H \in \mathcal{S} \setminus \{H_1, H_2\}$ be an arbitrary one of the halfspaces. Assume that $R(\mathcal{S} \setminus \{H\}) \neq \emptyset$. Write down a deterministic algorithm that runs in time linear in $n = |\mathcal{S}|$ and that on input $(\mathcal{S}, H, c(\mathcal{S} \setminus \{H\}))$ determines whether $R(\mathcal{S}) \neq \emptyset$ and if so outputs $c(\mathcal{S})$.

(c) Let again $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above). Using (b), write down a randomized algorithm which, given $\mathcal{S}$, determines whether $R(\mathcal{S}) \neq \emptyset$ and if so outputs $c(\mathcal{S})$. Your algorithm should run in expected time linear in $n = |\mathcal{S}|$.

[1] this rotation can always be done such that we also do not have vertical or horizontal lines, which you may assume

In-Class Exercise 5: Minimum Cut

(a) Let G be the multigraph with vertices $1, \ldots, 12$ depicted below, and let e be an edge of G chosen uniformly at random. Calculate the value of $\Pr[\mu(G) = \mu(G/e)]$.



(b) Let $\mathcal{A}$ be a randomized algorithm which, when given any multigraph G on n vertices as input, outputs a certain (random) value A, where $A = \mu(G)$ with probability $1/n$ and $A < \mu(G)$ in all other cases. Let $A_1, \ldots, A_k$ be the values returned by k independent invocations of $\mathcal{A}$ with input G. Choose a specific value for k and describe another algorithm $\mathcal{B}$ which takes $A_1, \ldots, A_k$ as input and which outputs a value B such that $B = \mu(G)$ with probability at least $1 - n^{-42}$.